First Year Report

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7^{th} October 2002

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§ 1. Introduction

§ 1.1 Geometry and mathematics

The simplest geometrical figure is a circle while the simplest of all polygons is a triangle. The degree of freedom of triangles increases from the equilateral to the isosceles and the right triangles to the scalene triangles. While spending the summer of 1990 in a traineeship through AIESEC I was introduced to a geometrical puzzle which, as I came to learn later, is called the *flexatube*, but which I conjectured at the time that was from ancient China. This puzzle is made up of sixteen right isosceles triangles tiled into four squares, each comprising of four triangles, which are in turn joined together to form a loop. It can be easily made up using some hard papers, a pair of scissors and cello tape. There are in total twenty hinges, four of which are as long as the hypothenuse while the other sixteen have their length equal to the shorter side of the triangle. By turning these rigid triangles upon their hinges the inside surface of the strip can become the outside and vice versa. I found one solution a week later and back in Thailand a friend of mine found another solution. I think that these two are the only possible solutions but have not been able to proof it.

Having this interest in geometrical puzzles I was delighted at the time to find that the symbol for the men's toilets in Budapest is an equilateral triangle, while that for women's toilets is a circle. No other things beside these are written. Obviously these two geometrical forms suffice and are fully understood by the whole population.

In general dimension we talk about spheres. A sphere in d dimensions has its volume, V, proportional to r^d and its surface, A, to r^{d-1}, so that $V \propto A^{\frac{d}{d-1}}$.

The area of sphere in three dimensions is $A = 4\pi r^2$ and the volume $V = \frac{4}{3}\pi r^3$. When $V = 1$,

$$\begin{aligned} r &= -\frac{\left(-\frac{3}{\pi}\right)^{\frac{1}{3}}}{2^{\frac{2}{3}}}, \frac{\left(\frac{3}{\pi}\right)^{\frac{1}{3}}}{2^{\frac{2}{3}}}, \frac{(-1)^{\frac{2}{3}}\left(\frac{3}{\pi}\right)^{\frac{1}{3}}}{2^{\frac{2}{3}}} \\ &= -0.310175 - 0.537239 \subset, 0.62035, -0.310175, -0.310175 + 0.537239 \subset . \end{aligned}$$

Therefore $\alpha = \frac{A}{V|_1} = 6^{\frac{2}{3}}\sqrt[3]{\pi} = 4.83598$

When $V = 8$, $r = \left(\frac{6}{\pi}\right)^{\frac{1}{3}}$ and therefore $A = 46^{\frac{2}{3}}\pi^{\frac{1}{3}} = 19.3439$. In order to find α, the surface area per unit volume, one divide the area by $V^{\frac{2}{3}}$, in other words $\alpha = V^{\frac{1}{3}} \cdot \frac{A}{V} = \frac{A}{V^{\frac{2}{3}}}$.

The same is true for other polyhedra. For example in a tetrahedron where x is the length of the side, the vertices can be $(0,0,0), (x,0,0), \left(\frac{x}{2}, \frac{\sqrt{3}}{2}x, 0\right), \left(\frac{x}{2}, \frac{\sqrt{3}}{6}x, \frac{1}{2}\sqrt{\frac{93}{35}x}\right)$. When $V = 1$, one can obtain x by solving the equation

$$1 = \frac{1}{6}\,\mathrm{abs}\left(\begin{vmatrix} 0 & 0 & 0 & 1 \\ x & 0 & 0 & 1 \\ \frac{x}{2} & \frac{\sqrt{3}}{2}x & 0 & 1 \\ \frac{x}{2} & \frac{\sqrt{3}}{6}x & \frac{1}{2}\sqrt{\frac{93}{35}}x & 1 \end{vmatrix}\right).$$

This gives $x = 2\left(\frac{35}{31}\right)^{\frac{1}{6}} = 2.0409$ as the only real positive answer. When x is doubled, V increases from 1 to 8, which means that one would be dividing A by $V^{\frac{2}{3}}$ to obtain α.

The perimeter of a triangle is $P = 3l$ and the area $A = \frac{\sqrt{3}}{4}l^2$. When $A = 1$, $l = \pm\frac{2}{\sqrt[4]{3}} = \pm 1.51967$. Therefore $\prime = 4.55901$

Vertices shared by two cells make up a common face between them. Two way have been tried for finding the edges. The first one was by looking at all neighbouring cells of every cell in turn three at a time. The edges are then made up of those vertices that are common among these three cells. Only those edges which have exactly two vertices are considered. They are called *good edges* as contrasted with edges on the boundary. This is a much longer way than the second one, which is to consider vertices common to any two faces of a cell. Similar to the first case, such vertices forms a good edge if and only if there are only two of them. The two methods above give exactly the same list of edges, so they confirm each other. It has been tested that all edges having more than two vertices are boundary ones, that is they have at least one vertex outside the boundary of the unit cube considered.

By drawing some of the cells as a solid using *fill* command it has been tested that the result from *convhull* covers the entire cell surface. This confirms the step where areas are calculated.

The hexagon or honeycomb is perhaps the pattern which is most frequently found in nature. Eventhough the world we live in is three-dimensional, cells normally divide and spread in two

dimensions in the form of layers. Moreover, they are packed in these layers in patterns which most often resemble the honeycomb (*cf* Williams and Bjerknes, 1972).

An octagon is an eight-sided polygon. It is the shape of the cross section of every chimney in the mills built in Manchester during its industrial era of the nineteenth century, as well as that of the terrets in the Main Building of UMIST. Perhaps one of the reasons for its popularity is that it looks strong while having the style of a good taste. May be the reason why it looks strong is that it possesses eight axes of symmetry, on top of another symmetry around the origin.

There are nine regular polyhedra. Among these are five regular convex solids known to the ancient Greek called Platonic polyhedra. They are tetrahedron, cube, dodecahedron, octahedron, and icosahedron. They have regular congruent faces and regular polyhedral angle vertices. Their face angles and their dihedral angles at every vertex are equal. The other four regular polyhedra have only been discovered much later and are not convex. They are called the Kepler-Poinsot polyhedra and are nonconvex. The small stellated dodecahedron and the great stellated dodecahedron were found by Kepler (1571–1630). The great icosahedron and the great dodecahedron were found by Poinsot (1777–1859). The small stellated dodecahedron and the great dodecahedron do not satisfy Euler's equation. The process of creating it by extending nonadjacent faces until they meet is called *stellating.* There are also polyhedra called quasi-regular.

The semi-regular polyhedra are called the Archimedean polyhedra. Here all faces are regular polygons but not all are of the same kind. Every vertex is congruent to all others. They comprise of an infinite group of prisms, an infinite group of antiprisms or prismoid, and another thirteen polyhedra. Each prism or prismoid is made up of two regular polygons on parallel planes where the vertices are aligned in the former case or shifted half way to the next neighbouring vertices in the latter case. Each vertex in prisms is joined to a corresponding vertex of the opposite polyhegon, while in prismoid it is joined to two corresponding vertices. All faces of an Archimedean solid are regular and all its polyhedral angle vertices congruent.

On the other hand the Archimedean duals have got the property that all their faces are congruent to one another and all their polyhedral angles regular. These solids are important in crystallography. They are vertically regular. They include an infinite group of dipyramids, an infinite group of trapezohedra, and additionally thirteen other polyhedra.

The surface area per unit volume α of a solid can be computed from the actual volume V and the actual surface area A as $\alpha = \sqrt[3]{V}\frac{A}{V} = AV^{-\frac{2}{3}}$.

Polygon	*Edges*	*Area*	$\wp$	$\wp$ *(numerical)*
Triangle	3	$\frac{\sqrt{3}}{4}l_e^2$	–	4.55901

Surface area, volume and α of some solids are shown in the following table.

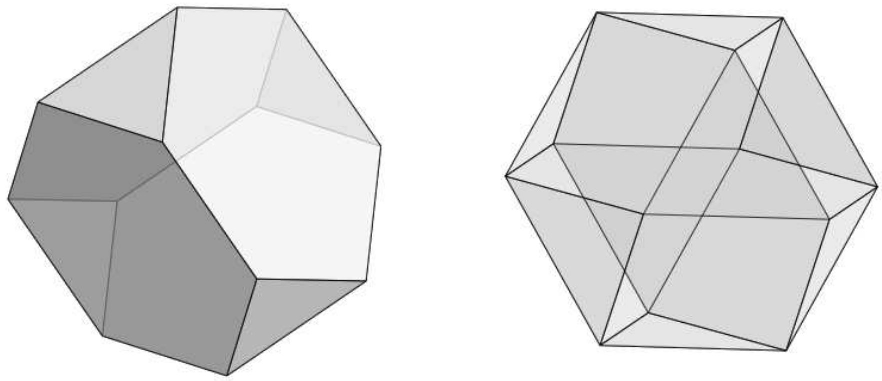
Solid	*Surface area*	*Volume*	α	α *(numerical)*
Cube, (4^3)	6			
Dodecahedron,(5^3)				
Icosahedron, (3^5)				
Octahedron, (3^4)				
Sphere	$4\pi r^2$	$\frac{4}{3\pi r^3}$	$6^{\frac{2}{3}}\sqrt[3]{\pi}$	4.83598
Tetrahedron (3^3)	7.2056			

The tetrahedron is self-dual. The octahedron is dual to the cube while the dodecahedron the icosahedron.

The nearest neighbour and minimum spanning tree have been applied to the problem of taxonomy in botany. Clayton (1972), working on the characters of plants to manually classify them (*eg* Clayton, 1970) with the use of only the binary dendrogram and trial and error, adopted a numerical method which finds the minimum spanning tree in a multi-dimensional character space. Since taxonomy can be considered as a kind of dictionary, it is possible to apply a similar approach to machine translation and the compilation of dictionaries.

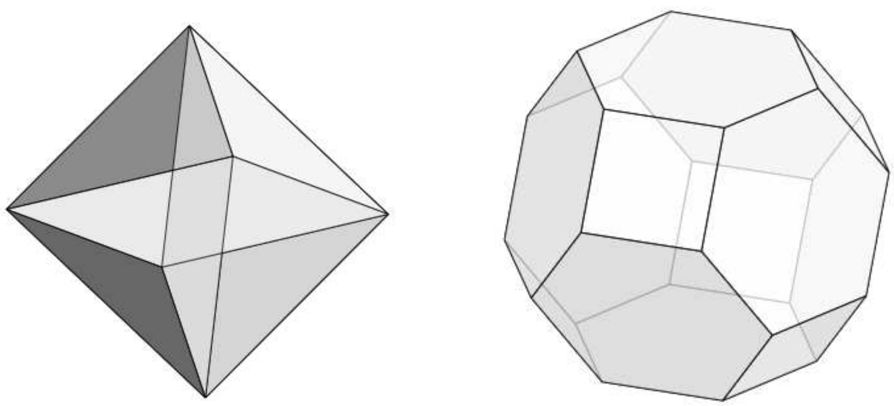
The icosahedron is a common shape found among viruses.

The polyhedra from Figure 1.1 to 1.3 are semi-regular.



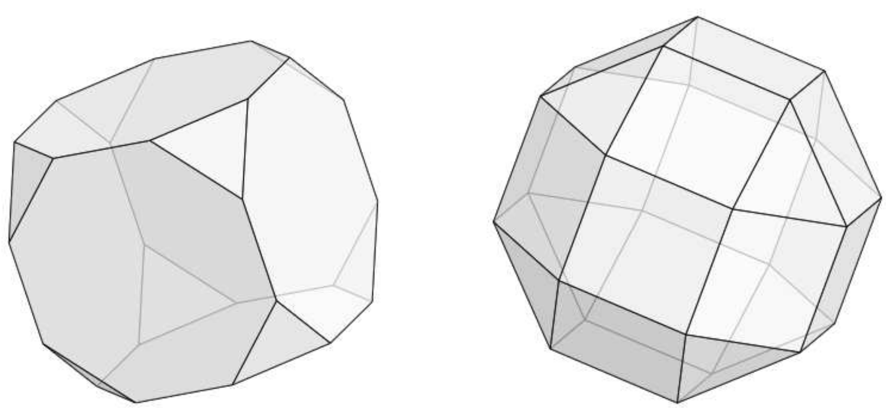
(a) (b)

Figure 1.1 (a) *Truncated tetrahedron, triakistetrahedron,* 2 3|3. (b) *Octahemioctahedron, octahemioctacron,* 3/2 3|3



(a) (b)

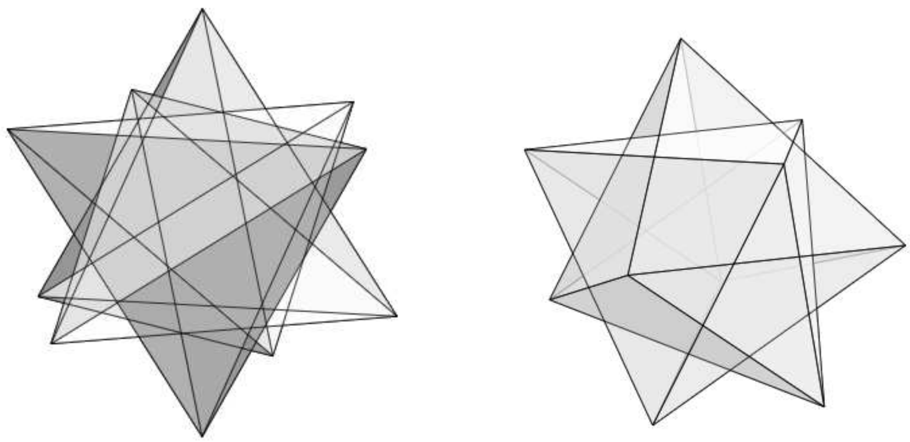
Figure 1.2 (a) *Tetrahemihexahedron, tetrahemihexacron,* 3/2 3|2. (b) *Truncated octahedron, tetrakishexahedron,* 2 4|3



(a) (b)

Figure 1.3 (a) *Truncated cube, triakisoctrahedron,* 2 3|4. (b) *Rhombicuboctahedron, deltoidal icositetrahedron,* 3 4|2

Polyhedra in Figure 1.4 are snub polyhedra.



(a) (b)

Figure 1.4 (a) *Pentagrammic crossed antiprism, pentagrammic concave deltohedron,* $|2\,2\,5/3$. (b) *Pentagrammic antiprism, pentagrammic deltohedron,* $|2\,2\,5/2$

The surface of Fullerine is made up of pentagons six-sided figures. Its shape represents that of the geodesic domes developped by Buckminster Fuller, and hence the name *Fullerine*. The latter may either be hexagons or figures all the six sides in each one of which form two sets of three sides having an equal length. The simplest Fullerine, the carbon-60 molecule, has the same shape as that of a football and a handball. With some thought the reason for this is not difficult to see. With its thirty-two faces it closely resemble the sphere. Also the two different shapes of all its components are symmetrically distributed and therefore enable colouring with only two different colours, namely one for each of the two shapes. To see how this helps, suppose one made a football in the shape of a bloated dodecahedron. Then it would be impossible to colour it using more than one colour at the same time of giving it a symmetrical appearance when viewed from more than a few directions. With the fullerine shape and the colouring scheme mentioned, however, the football looks symmetrical when viewed from 54 different directions symmetrically distributed around it. These directions corresponds to those when one looks at it in the direction perpendicular to the centre of each of its faces and when in the direction through the middle of each of the 22 edges lying between two hexagonal faces.

Making polyhedron models is an educating experience. Contrary to the general believe that you need to make an accurate drawing for the required parts (Wenninger, 1971), this needs not be so. Examples of this are the origami models of polyhedra where complex polyhedron structures are made from interlocking pieces each of which is made by folding a piece of paper of a rectangular or square shape.

A set of elements with the sum and the product of any two elements defined is a commutative ring if under these two operations it satisfies the following postulates: closure, uniqueness, commutative, associative, and distributive laws, identity (zero and unity), and additive inverse. An integral domain is an ordered domain if its positive elements satisfy the laws of addition, multiplication, and trichotomy. A subset of an ordered domain is well-ordered if every nonempty subset of it contains a smallest member. $a|b$ means that $.b$ is divisible by a. The Euclidean algorithm or division algorithm states that $a = bq + r, 0 \leq r < b$. Two integers are relatively prime if their only common divisors are ± 1. $a \equiv b (\bmod m)$ if and only if $m|(a - b)$. The commutative ring Z_2 is the properties of multiplication and addition of even (0) and odd (1) numbers.

+	**0**	**1**
0	0	1
1	1	0

·	**0**	**1**
0	0	0
1	0	1

The following is Z_5.

+	**0**	**1**	**2**	**3**	**4**
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

There is a close link between geometry and algebra. Geometrical surfaces can be described

as algebraical equations. For example, for circles and polygons the equations are binary quadratic, while for spheres and polyhedra they are ternary quadratic. Even one-sided surfaces can be described algebraically. The equation of Klein bottle, when deformed into a sphere with two circles removed and replaced by two cross-caps, is a quartic equation

$$a^2(x^2+y^2)(b^2-x^2-y^2) = z^2(a^2x^2+b^2y^2),$$

while the Steiner surface is also a quartic one

$$y^2z^2+z^2x^2+x^2y^2+xyz=0.$$

Two surfaces is homomorphic to each other if it is possible to continuously transform one into the other. All convex polyhedra are homomorphic to a sphere. The Steiner surface is homomorphic to the heptahedron, which is an Archimedean polyhedron with diametral plane.

In the plane, a second-degree equation gives either two straight lines, a circle, an ellipse, a parabola, or a hyperbola. In space, it can give two planes, cylinders and cones (circular, elleptic, parabolic, or hyperbolic), aphere, spheroid, ellipsoid, two hyperboloids, and (elliptic or hyperbolic) paraboloid.

Partition, *tessellation* and *division of space* are the same thing. In the context of set theory, a partition of set X is a family of sets A_1, A_2, …, A_k which are subsets of X, such that $A_i \neq \emptyset$; $A_i \cap A_j = \emptyset$; $\bigcup_i A_i = X$, where i, $j = 1$, 2, …, k and $i \neq j$. (*cf* Berge, 1958) A further condition that makes any tessellation a Voronoi one is that, for all i there exists a unique point a_i within A_i such that every point in A_i is closer to a_i than to any other a_j, $j \neq i$.

Voronoi tessellation in three dimensions can be constructed by imagining each region as a spherical cell growing outwards to meet neighbouring cells and continue growing to fill the gaps. The centre of each sphere is a unique nucleus point of the region such that it is closest to any point belonging to that region than any nuclei points. If the rate of growth is the same from every cell, the resulting partitions will be planes which can be described by ternary quadratic equations. However, if this rate differs from one cell to another, the partitions will be curved surfaces and the result is a non-Voronoi tessellation. It is possible to impose a constraint of minimum distance between neighbouring nuclei. Such cases can be looked at as spheres of an equal nonzero radius expanding away from nucleus centre points. If the radii differ from one sphere to another, or if some nonspherical solids are used instead of spheres, the tessellation obtained will be non-Voronoi.

Consider the case where all spheres are of equal size. If these spheres already touch their neighbours before the expanding starts, the case is that of packed spheres expanded to form a Voronoi tessellation. There are two types of close-packing: cubic (face-centred) and hexagonal. In both cases each sphere has twelve neighbours. Both cases have the same density, which is $\frac{\pi}{3\sqrt{2}}$. The Voronoi regions produced from the cubic case are rhombic dodecahedra and the faces are rhombuses. In the case of hexagonal close-packing, the corresponding regions are trapezo-rhombic dodecahedra and the faces are either rhombics or trapezia. Where the spheres meet with their three neighbours in the layer above and their three neighbours in the layer below, the faces are rhombics. Where they meet with the six neighbours on the same layer they are trapezia.

For geometrical calculation, an example of a definitive book is that written by Salmon (1912).

The gamma function,

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\mathrm{d}t, \quad \mathrm{Re}(z) > 0, \qquad (\mathbf{1})_{\mathrm{i}}$$

got its name from Legendre and is known as the Euler gamma function or simply the second Euler function. The formula $\Gamma(z+1) = z\Gamma(z) = z!$ recursively calculates the gamma function from, for instance, $\Gamma(1/5) \approx 4.5908$, $\Gamma(1/4) \approx 3.6256$, $\Gamma(1/3) \approx 2.6789$, $\Gamma(2/5) \approx 2.2182$, $\Gamma(1/2) = \sqrt{\pi} \approx 1.7725$, $\Gamma(3/5) \approx 1.4892$, $\Gamma(2/3) \approx 1.3541$, $\Gamma(3/4) \approx 1.2254$, and $\Gamma(4/5) \approx 1.1642$. The Stirling's formula was found by de Moivre which approximates the gamma function. The gamma function expansions is

$$\Gamma(x+1) = \lim_{k\to\infty} \frac{k^x 1\cdot 2\cdot 3\cdots k}{(x+1)(x+2)\cdots(x+k)}, \qquad (\mathbf{2})_{\mathrm{i}}$$

and the gamma function of negative numbers can be obtained from

$$\Gamma(-z) = \frac{-\pi}{z\Gamma(z)\sin \pi z}. \qquad (\mathbf{3})_{\mathrm{i}}$$

The incomplete gamma function is

$$\Gamma(z,x) = \int_0^x e^{-t}t^{z-1}\mathrm{d}t = \int_x^\infty e^{-t}t^{z-1}\mathrm{d}t, \qquad (\mathbf{4})_{\mathrm{i}}$$

and the normalised or regularised incomplete gamma function is $\Gamma(z,x)/\Gamma(z)$

§ 1.2 Statistics

Poisson distribution, defined by $p(x,\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}\mathrm{I}_{[0,\infty)}$, is the binomial distribution, $p(x,n,p) = {}^n\mathrm{C}_x\theta^x(1-\theta)^{n-x}\mathrm{I}_{[0,n]}$, when n goes to infinity, θ goes to zero, while $n\theta = \lambda$. Here θ is the probability of success of each trial. It is used when counting the number of occurences of a random event. Analogously Poisson point process, which has $p(x = n(v)) = \frac{\lambda|v|e^{-\lambda|v|x}}{x!}\mathrm{I}_{[0,\infty)}$, is the binomial point process, $p(x = n(v)) = {}^n\mathrm{C}_x\theta^x(1-\theta)^{n-x}\mathrm{I}_{[0,n]}$, when the volume V goes to infinity, while $\frac{n}{|V|} = \lambda$. Here $\theta = \frac{|v|}{|V|}$ is the probability of points within V being placed in $v \subset V \subset R^d$, and λ the density or intensity of points. Therefore the density of point of a Poisson point process is constant by definition. A point process is a procedure which generates points on a domain within a space of d dimensions.

The Poisson point process thus derived has the properties that $0 < p_{n(v)=0} < 1$ for $0 < |v| < \infty$, $\lim_{|v|\to 0} p(n(v) \geq 1) = 0$, $n(v_i)$ mutually independent and $n(\bigcup_n v_i) = \sum_n n(v_i)$ when A_i are disjoint, and $\lim_{|v|\to 0} \frac{p(n(v)\geq 1)}{p(n(v)=1)} = 1$.

The weighted mean of a group of data is $\mathbf{x} = \frac{\sum_i f_i x_i}{n}$ and the weighted variance is $\sigma^2 = \frac{\sum_i f_i(x_i-\mathbf{x})^2}{n}$, where f_i is the occurrence frequency of x_i and $\sum_i f_i = n$. Likewise the r^{th}-moment around the average is $m_r = \frac{\sum_i f_i(x-\mathbf{x})^r}{n}$, while the r^{th}-moment around the origin is $m'_r = \frac{\sum_i f_i x^r}{n}$ (*cf* Spiegel, 1975). Some relations among these various moments are $m_1 = 0$, $m_2 = m'_2 - {m'_1}^2$, $m_3 = m'_3 - 3m'_1m'_2 + 2{m'_1}^2$, and $m_4 = m'_4 - 4m'_1m'_3 + 6{m'_1}^2m'_2 - 3{m'_1}^4$.

The variance when normalised by $n-1$ gives the best unbiased estimated variance if the sample has a normal distribution. On the other hand the variance which is normalised by n is identical with the second moment of the sample about its mean.

§ 1.3 Phase transition

In the Ising model each spin has two possible states, that is up and down, and the hamiltonian is $H = J_0\sum_{<i,j>}\sigma_i\sigma_j$ where the summation is over the nearest neighbours. Since it has been exactly solved, the Ising model provides a good model for the understanding of phase transition. This model can represent the transition from ferro- to paramagnetic at the critical temperature where the correlation lenght becomes infinite. Characteristic to the Ising model is the peak in the specific heat at the critical temperature.

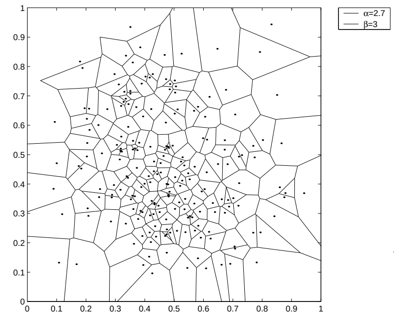
The two-dimentional xy model is a model of spins confined to a plane, the hamiltonian of which is $H = J_0\sum_{<i,j>}\cos(\theta_i - \theta_j)$. This model can represent the superconducting and the superfluid films. For this model there is no phase transition showing long-range ordering. One example is the two-dimensional Coulomb gas model where the vortex-antivortex pairs, which are bound to each other at low temperature, increases in number as the temperature increases and become separated at the KT temperature that marks the phase transition.

It had been generally believed that no phase transition can exist for the xy model when Kosterlitz *et al* (1973) showed that there is another kind of phase transition, arisen from the topological excitation of vortex-antivortex pairs instead of from the long-range ordering found in a spontaneous magnetisation. They consider the two-dimensional model of gas with charges $\pm q$ where the interaction potential is $U(|\mathbf{r}_i - \mathbf{r}_j|)$ is $-2q_iq_j\ln|(\mathbf{r}_i - \mathbf{r}_j)/\mathbf{r}_0| + 2\mu$ when $r > r_0$, and 0 when $r < r_0$. The problem is reduced to that of solving an equation of the form $\frac{(\mathrm{d}y}{\mathrm{d}x)=-e^{-xy}}$. The application mentioned there is in the xy model of magnetism, the solid-liquid transition, and the neutral superfluid, but not in a superconductor and a Heisenberg ferromagnet.

The frustrated xy model, the hamiltonian of which is $H = J_0\sum_{<i,j>}\cos(\theta_i - \theta_j - A_{ij})$, occurs when a magnetic field is applied perpendicular to the two-dimensional plane of the xy model. The frustration parameter, $f = \Phi/Phi_0$, is a measure of the average external magnetic flux. When $f = 1/2$ the model is called the fully frustrated xy model. The local chirality, $m(r_i) = \frac{1}{2\pi}\sum(\theta_i - \theta_j - A_{ij})$, which describes the property of the ground state, where it can either be $+1/ovr2$ or $-1/ovr2$. The network configuration at $\mathrm{T} < \mathrm{T}_c$ is that of a draught board, and has Z_2 symmetry. This regularity is broken by the formation of domain walls in an Ising phase transition at T_c.

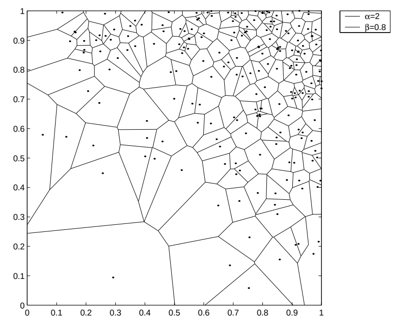
§ 1.4 Random processes

A synonym to *random* is *stochastic* (*cf* Miles, 1972). Any algorithm which employs a random element is called Monte Carlo. Random processes can have various types of distribution. The beta distribution has a probability density function $f_{X_{\alpha,\beta}}(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$, $0 \leq x \leq 1$, where $\alpha > 0$ and $\beta > 0$ are shape parameters, and $B(\alpha, \beta)$ is the beta function. There are three types of shape; the bridge shape has $\alpha > 1$ and $\beta > 1$, the J shape $\alpha \leq 1$ and $\beta \geq 1$, or $\alpha \geq 1$ and $\beta \leq 1$, and the U shape $\alpha < 1$ and $\beta < 1$.



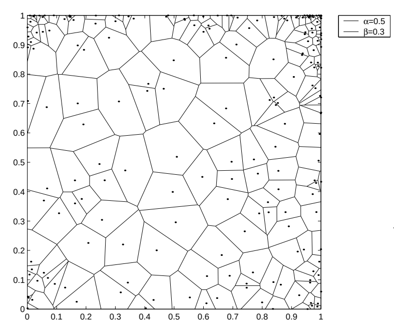
The 200 generators used in Figure 1.1 are randomly chosen with beta distribution with the shape parameters $\alpha = 2.7$ and $\beta = 3$, that is bridge shape.

Figure 1.1 Voronoi graph with bridge-shaped beta distribution.



Both x and y in Figure 1.2 have J-shaped distribution with the shape parameters $\alpha = 2$ and $\beta = 0.8$. The density is unbounded at $x = 1$ and at $y = 1$ because $\beta < 0$ for both.

Figure 1.2 Voronoi graph with J-shaped beta distribution.



The shape parameters in Figure 1.3 are $\alpha = 0.5$ and $\beta = 0.3$, that is U shape. The density is unbounded at $x = 0$, 1 and at $y = 0$, 1 because α is also less than zero.

Figure 1.3 Voronoi graph with U-shaped Beta distribution.

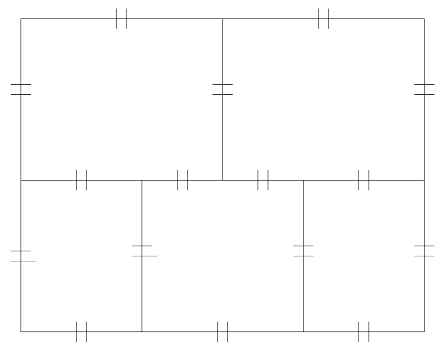
§ 1.5 Structures in nature

Prusinkiewicz and Lindenmayer (1990) model the structures in plants after having briefly discussed about the difference between the Chomsky grammars and the L-system, both of which being a mean for doing string rewriting, but the latter is based on the Turtle geometry which makes it convenient for the geometric rewriting of fractals. The states of a turtle consists of its position coordinates and the direction in which it is facing. Meinhardt (1995) models the patterns on sea shells by using mathematical based on the partial differential equations governing the system of activator, inhibitor, and substrate. Starting from a homogeneous initial condition, small deviations therein undergo a positive feedback and therefore increase. Activator catalyses both the production of itself and that of its inhibitor. The latter acts as a negative feedback which limits and makes the reaction local.

Voronoi tessellation

§ 2. Voronoi tessellation

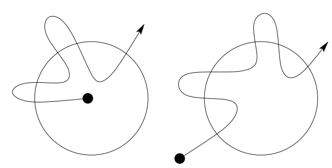
In a puzzle of Figure 2.1 there are five rooms with doors in the position as shown. The problem is whether one can walk through every door only once and the answer according to the graph theory is *no*, because there are more than two rooms which has an odd number of doors. The proof of Theorem 2.1 was from Komsan Bajàrạvàṇijỳ around 1989. From this, when one plays a puzzle like that of Figure 2.1 one always starts off from a room with an odd number of doors and ends in another such room. This is the same thing as saying that one starts and ends outside rooms with an even number of doors. Therefore the number of the latter is of no consequence, but that of the former is crucial for the existence of a solution and must never be any number other than two.



In Figure 2.1 there are three rooms with five doors, two with four, and one with nine. There are here four rooms with an odd number of doors. Starting off from one of these four one can only end up in one of the other three, which leaves the remaining two rooms unaccounted for. In other words at least two doors will necessarily remain unvisited.

Figure 2.1 Puzzle of five rooms with doors.

Theorem 2.1. *A travel along a network can only starts and ends at nodes which have an odd coordination number.*



The proof of Theorem 2.1. **Proof.:** Looking at Figure 2.2 if one starts from inside a room with an odd number of doors, one always ends up outside it. On the other hand one always ends up inside a room with an even number of doors if one starts of from it. In the second picture such a room is all the area outside the circle.

Figure 2.2 Departure and arival rooms.

□

The following Corollaries 2.1, 2.2 and 2.3 assume nondegeneracy of the Voronoi network. Such a path as mentioned in these corollaries is also called *self-avoiding*.

Corollary 2.1. *There can be no path which traverses all edges of a Voronoi graph only once.*

Proof.: This follows from Theorem 2.1 because a two dimensional Voronoi network has a coordination number three. □

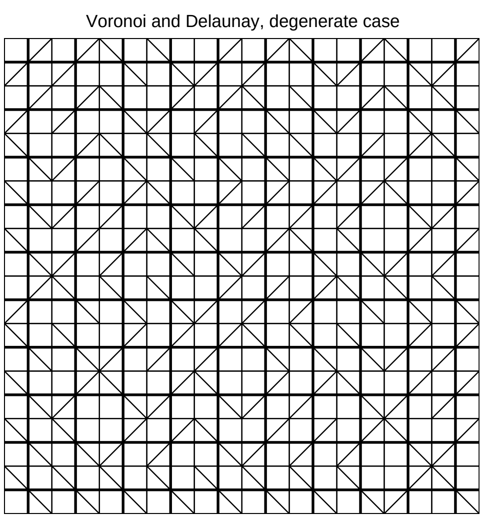
Corollary 2.2. *On a three dimensional Voronoi structure there always exists a path that runs through every edge once and only once.*

Proof.: This also readily follows from Theorem 2.1 since a three dimensional Voronoi network has a coordination number of four. □

Corollary 2.3. *Take any Voronoi cell of the three dimensional network, it is impossible to walk through all its edges without repeating some of them.*

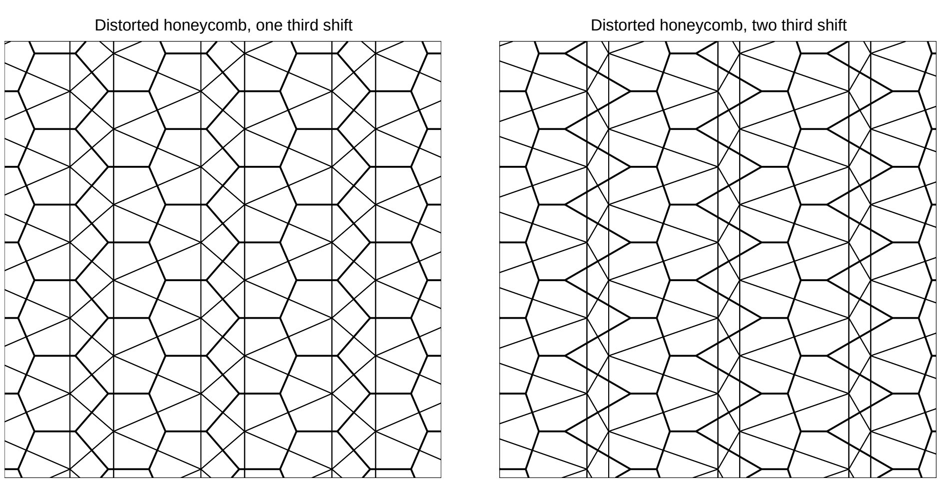
Proof.: Again from Theorem 2.1 and because the surface of a Voronoi polyhedron is a two dimensional network of polygons which has the coordination number three. □

Jerauld *et al* (1984) compared the Voronoi, with the triangular networks and found that the bond percolation probability of the former is 4.3% or 0.015 less than that of the latter, small site clusters more, and small bond clusters less likely.



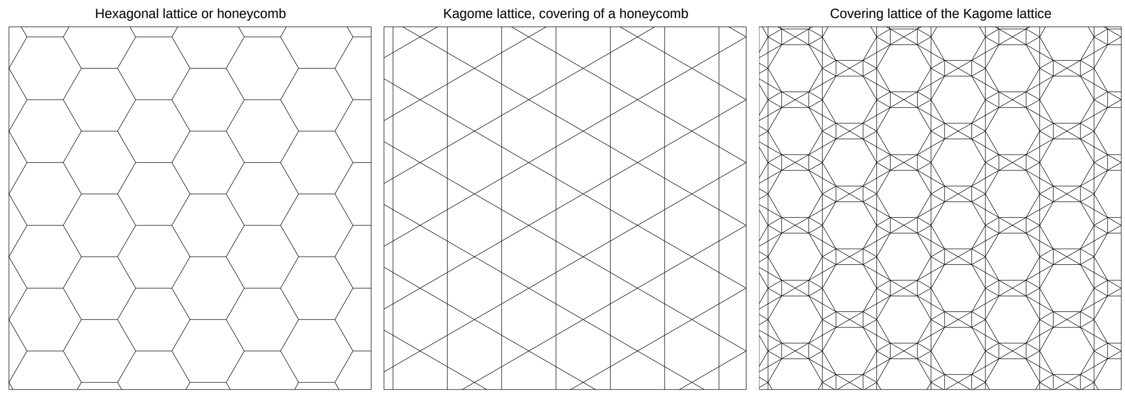
When a square lattice was fed to *voronoi* and *delaunay* in Matlab, by the program `degen.m` in § ℵ, there were error messages saying that points were collinear and possibly triangulation is incorrect. This case, Figure 2.3, is degenerative.

Figure 2.3 Voronoi from degenerative data.



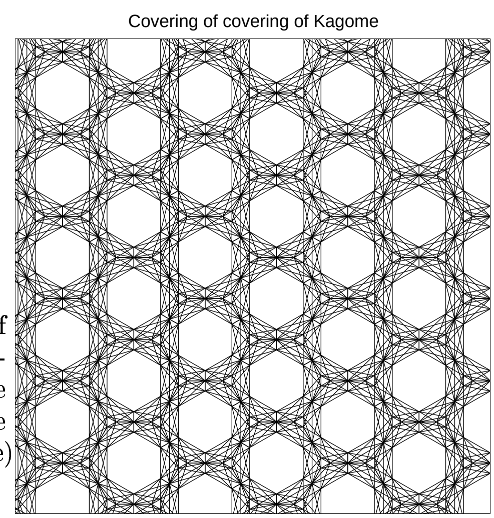
(a) (b)

Figure 2.4 *The honeycomb or hexagonal lattice whose alternate y-plane has been shifted (a) one-third, and (b) two-third respectively. Triangulation is shown with thinner lines. The program used is* `honey.m`



((a)) ((b)) ((c))

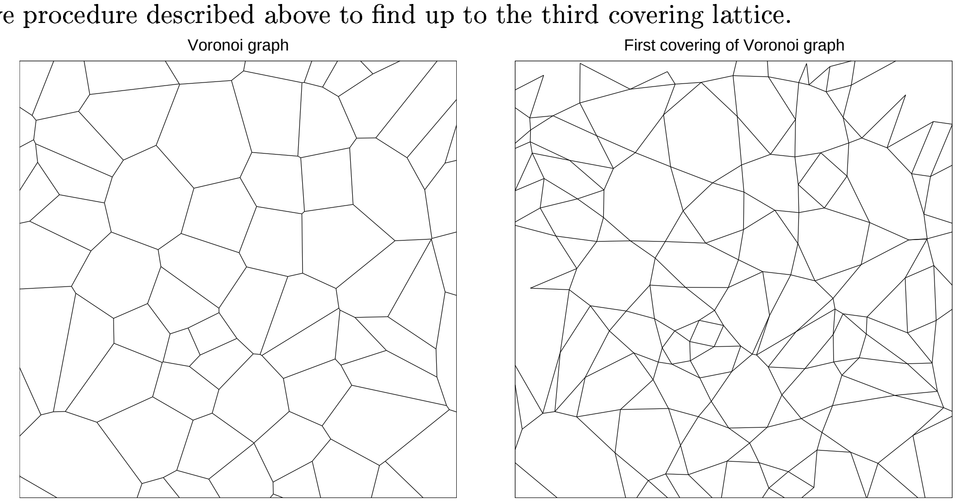
Figure 2.5 (a) *The hexagonal lattice,* (b) *its covering lattice (Kagome), and* (c) *the covering lattice of its covering lattice (i.e. covering of Kagome). (*`cover.m`, `covers.m`*, and* `coverss.m`*)*



If we indicate by $C_v^n(x)$ the n^{th}-order covering lattice of a lattice x, then the first picture is Hexagonal, the second one Kagome or C_v^1(Hexagonal) and the third one C_v^2(Hexagonal) or in other words C_v^1(Kagome). Figure 2.6 is the next iteration, a C_v^3(Hexagonal) or C_v^2(Kagome)

Figure 2.6 The next covering lattice.

Now let us look at the Voronoi graph and its covering lattices. Pictures in Figure 2.7 are drawn by first creating and cropping a Voronoi graph with the help of the program `crop.m`, then use the recursive procedure described above to find up to the third covering lattice.



(a) (b)

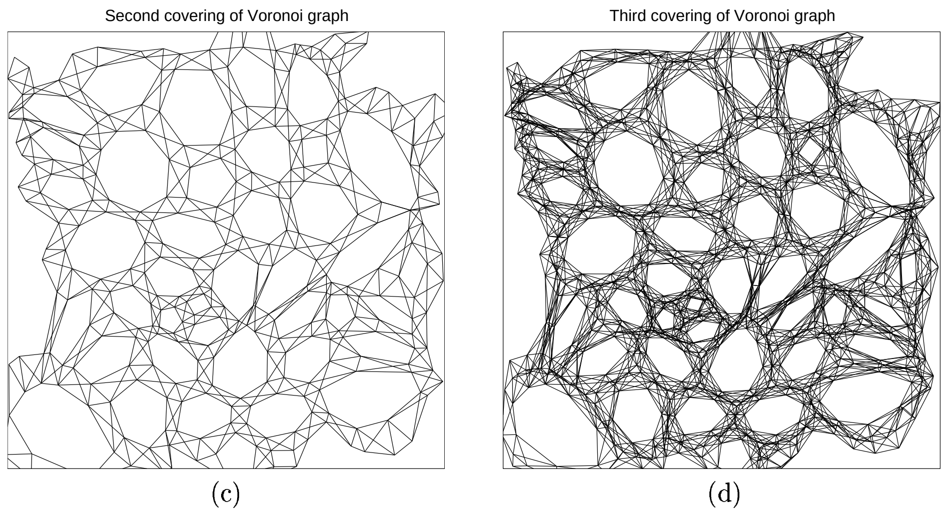


Figure 2.7 (a) *Voronoi graph (V.g.)*, (b) $C_v^1(\mathbf{V.g.})$, (c) $C_v^2(\mathbf{V.g.})$, (d) $C_v^3(\mathbf{V.g.})$.

Notice that the covering lattices retain the skeleton structure of the original Voronoi graph. Those of higher orders represent closer the structures of nature where walls have thickness.

§ 2.1 Quadratic equations

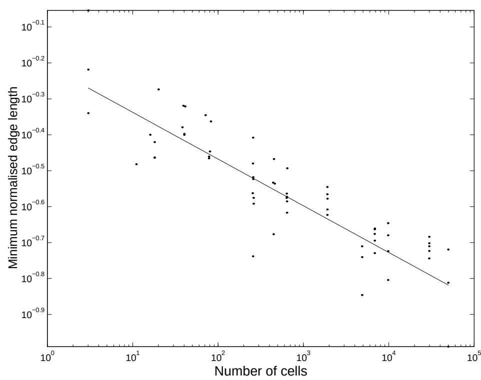
Quadratic equations are equations of binary quadratic forms. Around 400 BC Barbilonia had algorithmic equivalences of quadratic equations which are based on the method of completing the square and where all answers are unsigned, *i.e.* positive, lengths. Because there was no notion for zero, Diophantus considered three types of quadratic equations $ax^2 + bx = c$, $ax^2 = bx + c$, and $ax^2 + c = bx$. Euclid, *circa* 300 BC, used geometric equivalences or quadratic equations whose roots are also lenghts. Brahmagupta allowed negative quantities, which he called debts, and used abreviations for the unknown. Al-Khwarizmi classified quadratics into six types, namely squares equals roots, squares equals numbers, roots equal numbers, squares and roots equal number, squares and numbers equal roots, and roots and numbers equal squares. In his book *Liber embadorum*, published in 1145, Abraham bar Hiyya Ha-Nasi (aka Savasorda) gives the complete solution of quadratic equations. Luca Pacioli published *Summa de arithmetica, geometrica, proportioni et proportionalita* (or Summa) in 1494. He also applied quadratic methods to quartics of the form $x^4 = a + bx^2$. Scipione del Ferro solved the cubic equations of the form $x^3 + mx = n$.

§ 2.2 Quadratic forms

The theory of quadratic forms and the theory of matrix are inseparable though the history of these two subjects are somewhat fragmentary. A bilinear form in the sets x_i and y_i, $i = 1 \ldots n$, is $\sum_{i,j} x_i y_j$, or $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{y}$. If $x_i = y_i$ for all i, then the form is quadratic in x_i. In other words, a quadratic form is a general expression which contains second order terms.

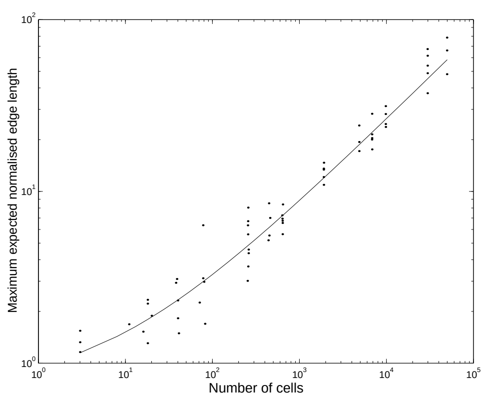
§ 2.3 Voronoi statistics

In two dimensions the statistical descriptions are as follows.



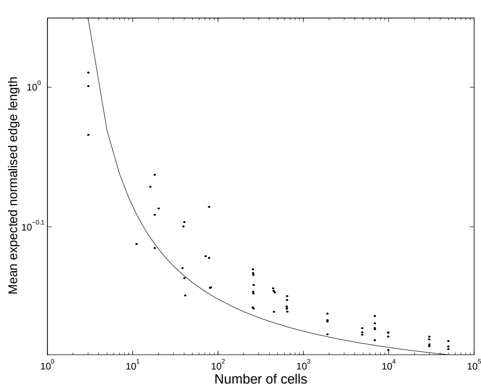
First the edge lengths are normalised by the edge length of the equivalent or characteristic square . A *equivalent square* is defined as the square figure whose area is equal to the area of the polygon in question, here a Voronoi polygon. Then find the average of the edge lenghts in each cell.

Figure 2.1 The minimum of the average normalised edge length. The curve is $y = \frac{0.62}{n^{0.13}}$.



Of these cell-averaged normalised edge length obtained from simulations on various sizes of networks the minimum values are plotted in Figure 2.2, the maximum in Figure 2.2, and the expected value in Figure ℵ. Note that the last quantity is the average over the whole structure of all the averages obtained one from each cell.

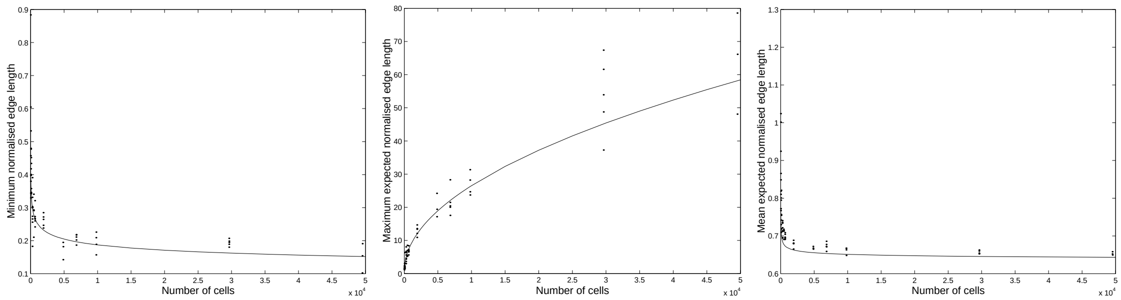
Figure 2.2 The maximum of the average normalised edge length. The curve is $y = \left|\sqrt{\frac{n}{15}}\right| + 0.7$.



To summarise, as the networks gets larger its minimum, maximum, and mean of the edge lenghts when compared with the characteristic length approach constant values. The characteristic length is defined as $l = \sqrt{\frac{\sum_{i=1}^{n} A_i}{n}}$, where A_i is the area of the i^{th} polygon and n is the number of cells.

Figure 2.3 The mean of the average normalised edge length. The curve is $y = 10^{\frac{0.2}{\log x}} - 0.4$.

That the three values mentioned become constant may not seem obvious by the look of Figures 2.3 to 2.3 because the scale used there is a logarithmo-logarithmic scale, not a Euclidean one. These figures emphasise the smaller ranges of size. Figure 2.4 below on the other hand is plotted using a normal scale which enables one to see the asymptotic effect more clearly. Here the figures (a), (b), and (c) are respectively Figure 2.3, 2.3, and *manel.* Let the term *representative* stands for ‘of the cell-average normalised’, and *length* means ‘edge length’ in this context. Then the *minimum representative length* approaches 0.15 from Figure ℵ, the *maximum representative length* is ever increasing, seemingly by a power law of approximately 0.5, while *mean representative length* approaches the value of 0.65. Properties of the Voronoi tessellation can be divided into individual and collective properties. With this in mind the term *length* above represents a property, *representative* means individual, and *minimum*, *maximum* and *mean* show the collective attributes.



(a) (b) (c)

Figure 2.4 *Mean, maximum, and mean of the cell-average normalised edge length.*

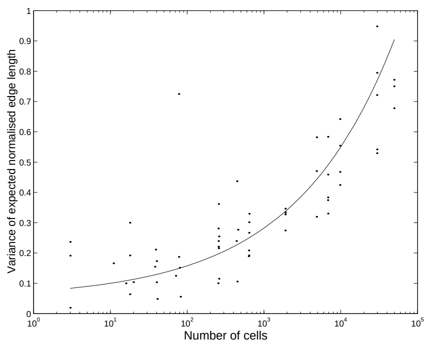
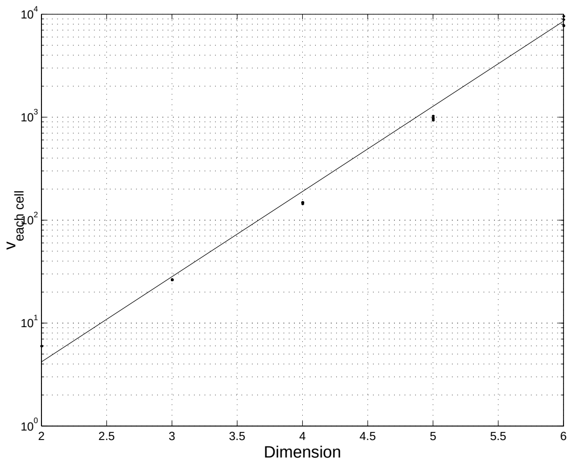
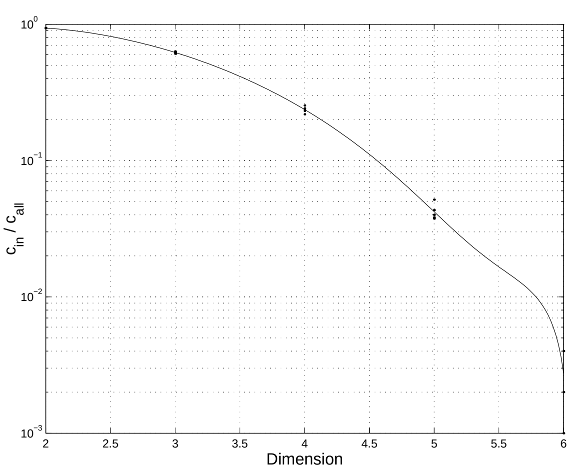


Figure 2.5 The variance of the expected values of the normalised length of edges of a cell. This $\sigma^2(\mathcal{N}_v^c(E(l_e)))$ increases very slowly with the increasing sizes of the networks. The curve shown has the equation $y = \left|\sqrt[3]{\frac{x}{8\times10^4}}\right| + 0.05$.



The number of vertices per cell increases dramatically as one goes up the dimension ladder. The program in § 3.9 contains the essential part of the code which produces Figure ℵ. The straight line shown is $0.093(4+e)^n$. Notice the trend towards a greater rate of increase at dimensions higher than the maximum six shown. Only Voronoi cells which lies within the original domain and the vertices of these are considered. The same program also gives Figure ℵ.

Figure ℵ Number of vertices per cell.



The expanse and the hypervolume of the Voronoi tessellation increases at an enormous rate, which results in the number of cells totally bounded within the original domain decreasing rapidly in Figure ℵ as the dimension goes up. The effect at close to the zero ratio is emphasised by using the log scale for the y-axis. Also with the logarithmic scale the polynomial estimation curves that give negative values of the ratio is automatically excluded.

Figure ℵ Ratio of cells in the original domain.

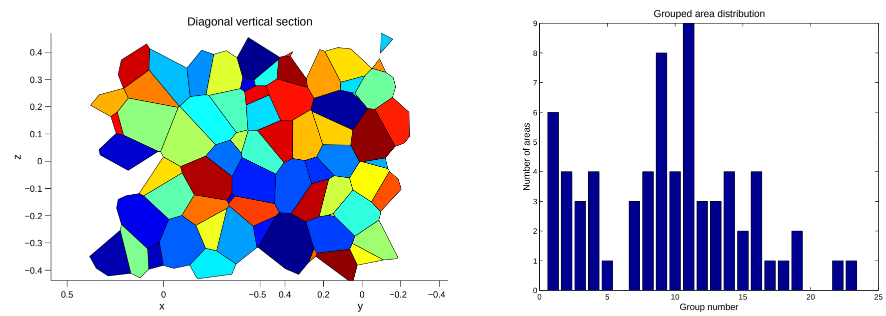
One can fit the polynomial $p(x) = p_1x^n + p_2x^{n-1} + \ldots + p_nx + p_{n+1}$ to the data with a least square algorithm. If $\mathbf{x}$ is the vector containing the data, then the the $(n+1)$ coefficients of the estimated polynomial can be found from $\hat{\mathbf{x}} = (\mathbf{x} - \mathrm{E}(\mathbf{x}))/\sigma(\mathbf{x})$. For data containing independent normal errors with a constant variance, the error bounds contain at least half of the predictions. The curve shown in Figure ℵ is $p(x) = -0.048x^4 + 0.064x^3 - 0.203x^2 - 0.459x + 0.263$, the average

value $\mathrm{E}(\mathbf{x})$ is 3.917, and the standard deviation $\sigma(\mathbf{x})$ is 1.412. The structure of the polynomial fit can be described using the Cholesky factor of the Vandermonde matrix

$$R = \begin{bmatrix} -12.19 & -1.56 & -6.08 & -0.47 & -3.17 \\ 0 & 8.46 & -0.45 & 4.47 & -0.44 \\ 0 & 0 & 1.21 & 0.33 & 2.93 \\ 0 & 0 & 0 & -1.63 & 0.30 \\ 0 & 0 & 0 & 0 & 2.25 \end{bmatrix},$$

the degree of freedom which is 19, and the norm of the residuals which is 0.034 in this case.

§ 2.4 Voronoi section



(a) (b)

Figure 2.1 *(a) Section by the plane $x - y + \epsilon z = \epsilon$, $\epsilon = 10^{-4}$. (b) Grouped distribution of area, the number of groups is approximately one-third the number of regions.*

	$\mathcal{V}_c$	$\aleph_c$	A_c^{fr}	V_c^{fr}	n_c^e	$E(n^e_{f,c})$	$\sigma(n^e_{f,c})$
min	8	6	2.2191×10^{-4}	9.9722×10^{-5}	12	4	0.6030
max	46	25	4.7015×10^{-3}	0.14364	69	5.52	2.5690
μ	26.338	15.169	1.8975×10^{-3}	1.8975×10^{-3}	39.507	5.1712	1.5586
σ^2	40.608	10.152	5.2144×10^{-7}	4.2313×10^{-5}	91.368	0.035983	0.1186
σ	6.3725	3.1862	7.2210×10^{-4}	6.5048×10^{-3}	9.5587	0.18969	0.3444
μ_g	25.543	14.829	1.7572×10^{-3}	1.1575×10^{-3}	38.315	5.1676	1.5169
μ_h	24.703	14.478	1.6026×10^{-3}	8.5548×10^{-4}	37.054	5.1638	1.4700
med	26	15	1.8125×10^{-3}	1.1660×10^{-3}	39	5.2	1.5706
mad	5.0068	2.5034	5.6975×10^{-4}	1.4250×10^{-3}	7.5103	0.14465	0.2715
$\mathcal{M}^2$	40.531	10.133	5.2045×10^{-7}	4.2233×10^{-5}	91.195	0.035915	0.1184
$\mathcal{M}^3$	72	9	2.5644×10^{-10}	5.4435×10^{-6}	243	-8.3404×10^{-3}	-0.0060
$\mathcal{M}^4$	5088.9	318.06	1.0467×10^{-12}	7.6653×10^{-7}	25763	7.9566×10^{-3}	0.041906
$\mathcal{K}$	3.0978	3.0978	3.8644	429.77	3.0978	6.1689	2.9916

Table 2.1 *Simulation uses rbox (1000 random points, seed 234985) and qhull (option v and o); $d = 3$, $n^c = 527$, $n^v = 6357$, CPU time 6,466.99 sec for the counting of statistics, 270.56 sec for finding area of the faces, 4.47 sec for calculating cell volume and 2.8 sec for finding the number of edges.*

§ 2.5 Number of vertices and edges

It has been observed from the simulations that in three dimensions cells always have vertices in even numbers and edges odd ones. This can be explained by the following theorems.

Theorem 2.1. *(cf Miles, 1972) In a simple three dimensional Voronoi tessellation, $3n_c^v = 2n_c^e$.*

Proof.: Pick any Voronoi cell of the tessellation. Suppose that it has n^v vertices. Add up the number of edges connected to all vertices. Because every cell is a simple polyhedron, there are exactly three edges connected to each vertex. The number of edges thus counted is therefore $3n^v$. But each of the edges is connected to two vertices, so we have counted every one of them twice. Therefore,

$$2n^e = 3n^v.$$

This is the case for any cell, hence the theorem is proved. □

This theorem gives rise the following two theorems.

Theorem 2.2. *The number of vertices of any cell within a simple three dimensional Voronoi tessellation is an even positive integer.*

Proof.: Observe that the term $2n_c^e$ in the theorem above is divisible by 2. This term is equal to $3n_c^v$, therefore the latter is also divisible by 2. Since 2 can not divide into 3, the only term left, n_c^v, must be divisible by 2 and hence even number. □

Theorem 2.3. *The number of edges of any cell within a simple three dimensional Voronoi tessellation is a positive integer divisible by three.*

Proof.: With the same line of reasoning as above, observe that $3n_c^v$ is divisible by 3. Therefore $2n_c^e$, and hence n_c^e, is also divisible by 3. □

Another proof for both the above theorems is the following.

Proof.: For an equality to hold, both sides must have the same factors. By supposing an unknown common factor i and by cross-multiplying the coefficients on both sides, one obtains $2\cdot(3\cdot i) = 3\cdot(2\cdot i)$, where i is a positive integer. Therefore $n_c^e = 3i$ and $n_c^v = 2i$. In other words, n_c^e is divisible by three and n_c^v is even. □

The theorems above assume that every cells are simple. This can not be the case in real situation where edges have dimensions and rather represent tubes than one-dimensional lines. Such case is similar to the so-called *degenerative* case in a computational model of Voronoi tessellation where there exist vertices the number of edges connected to each of which exceeds four. Even in the degenerative case, one would perhaps still expect a tendency for n_c^v to be an even number and for n_c^e to be divisible by three to hold.

In nature there are things which have a tendency towards even numbers. The following graph shows the the abundance in cosmic materials from the compilation by Cameron (1973). The year of publication of this paper is often misquoted as 1970, due to misprints in a footnote on the first page of the paper. Even the legendary Fred Hoyle has consistently practised this mistake and let it go uncorrected throughout his career. From what I have come across without an effort of searching, in two of his books and at least one of his papers, spanning the period of roughly fourty years in total (*cf* Hoyle, 1977). Out of a sample of 278 of those papers which cite this work, this misprint resulted in 80% of the total number of errors in the year cited, which in turn amounts to 1.8% of the number of samples.

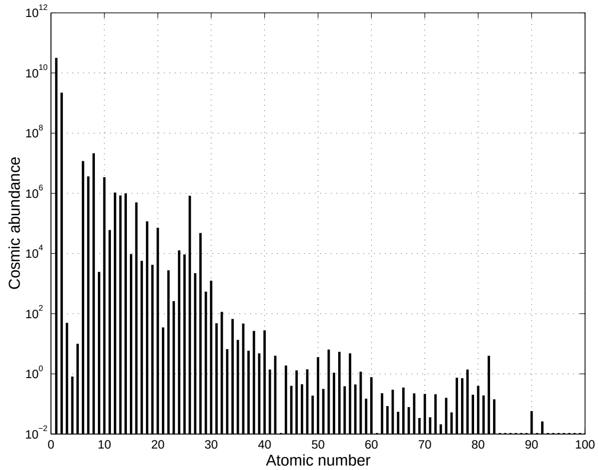
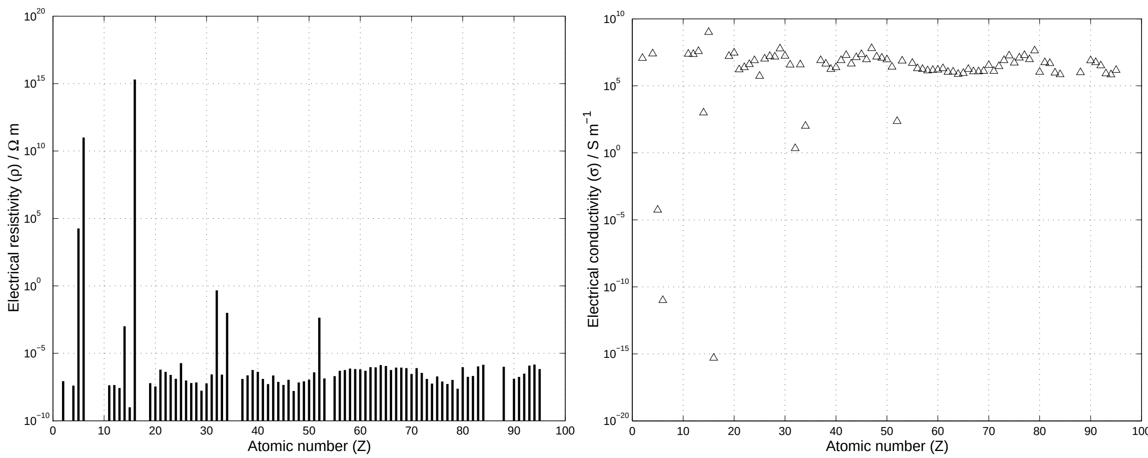


Figure 2.1 *The abundances of the chemical elements* in the universe. They are assumed to be the same as those found in the primitive solar nebula, which have been deduced from data on abundances found in chondritic meteorites and those found in the Sun. All abundances are relative to that of Si which is taken to be 10^6. Missing bars appear where the atomic numbers are unstable. Except for the atomic number 1 of Hydrogen, which is the most universal element, all other elements with even atomic numbers are locally more abundant than those with near-by odd atomic numbers.

Of interest are also the electrical resistivity and conductivity of solid matters.The conductivity σ is by definition the reciprocal of the resistivity ρ. Some of the solids, particularly boron, carbon, silicon, sulphur, germanium, selenium and tellurium, have a distinctively higher resistivity than the majority. Interestingly all of these, with only one exception of boron whose atomic number is five, are of an even atomic number, which respectively from carbon are 6, 14, 16, 32, 34, and 52. This can be seen in Figure 2.2 the data of which are taken from Podesta (2002).



(a) (b)

Figure 2.2 *The electrical resistivity, (a), and conductivity, (b), of elements which are solid at the room temperature.*

	n_c^e	$\mathcal{N}_v^c(E(l_c^e))$	$\mathcal{N}_v^c(\wp_c)$	$\mathcal{N}_v^c(A_c)$
min	3	0.20716	0.2749	0.058428
max	11	8.1072	10.134	22.307
μ	5.8973	0.70626	1.0089	1
σ^2	1.8955	0.18607	0.2942	1.4379
σ	1.3768	0.43136	0.54241	1.1991
μ_g	5.742	0.66036	0.94795	0.79225
μ_h	5.5904	0.62941	0.89999	0.61799
med	6	0.65709	0.96614	0.84267
mad	1.0736	0.17995	0.24357	0.49216
$\mathcal{M}^2$	1.8912	0.18565	0.29355	1.4347
$\mathcal{M}^3$	1.6194	0.96371	1.7931	22.703
$\mathcal{M}^4$	12.438	6.8376	15.785	468.03
$\mathcal{K}$	3.4775	198.38	183.18	227.39

Table 2.1 *Neighbour statistics. Here for the normalisation purpose,* $l^e_{\text{basis}} = 0.046662$, $\wp_{\text{basis}} = 0.18665, A_{\text{basis}} = 0.0021773$. *Simulation uses voronoin command in Matlab;* $d = 2$, $n^c = 448$, $n^v = 946$, *CPU time 1.19 seconds.*

Next simulation was done with $d = 2$, $n^c = 3$ to 49551.

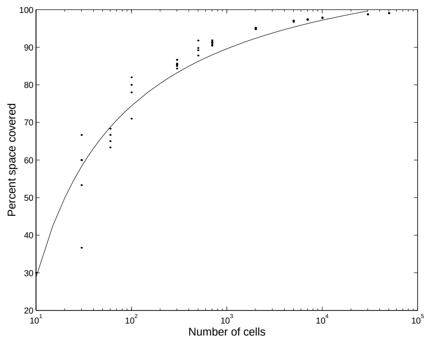
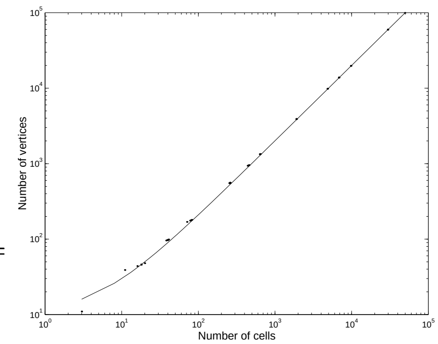
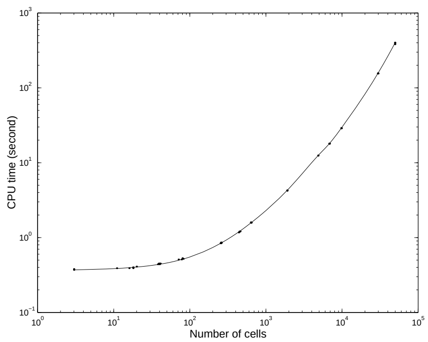


Figure 2.3 Percentage of space covered by a Voronoi structure. The number of cells is the total number of cells generated. The percent space covered is the volume of the structure after boundary cells, that is cells which extrude the unit volume boundary, have been excluded. The equation of the reference curve is $y = -\frac{210}{\log x} + 120$.



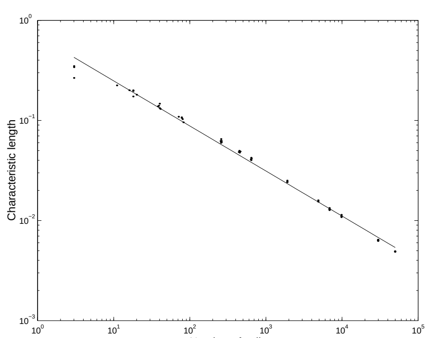
The reference line in Figure 2.4 is the linear equation $y = 2x + 10$. Boundary cells have been excluded.

Figure 2.4 Number of vertices versus number of cells.



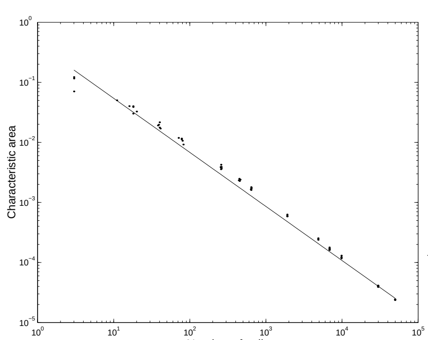
The curve in Figure 2.5 is the result of curve fitting by cubic spline interpolation.

Figure 2.5 The CPU time in seconds.



The characteristic length is the length of the side of the cubic structure having the same number of cells and the same total volume as the Voronoi structure. The characteristic lenghts in Figure 2.6 are shown as dots. The reference line is $y = \frac{0.7}{x^{0.45}}$.

Figure 2.6 Characteristic length versus the number of cells.

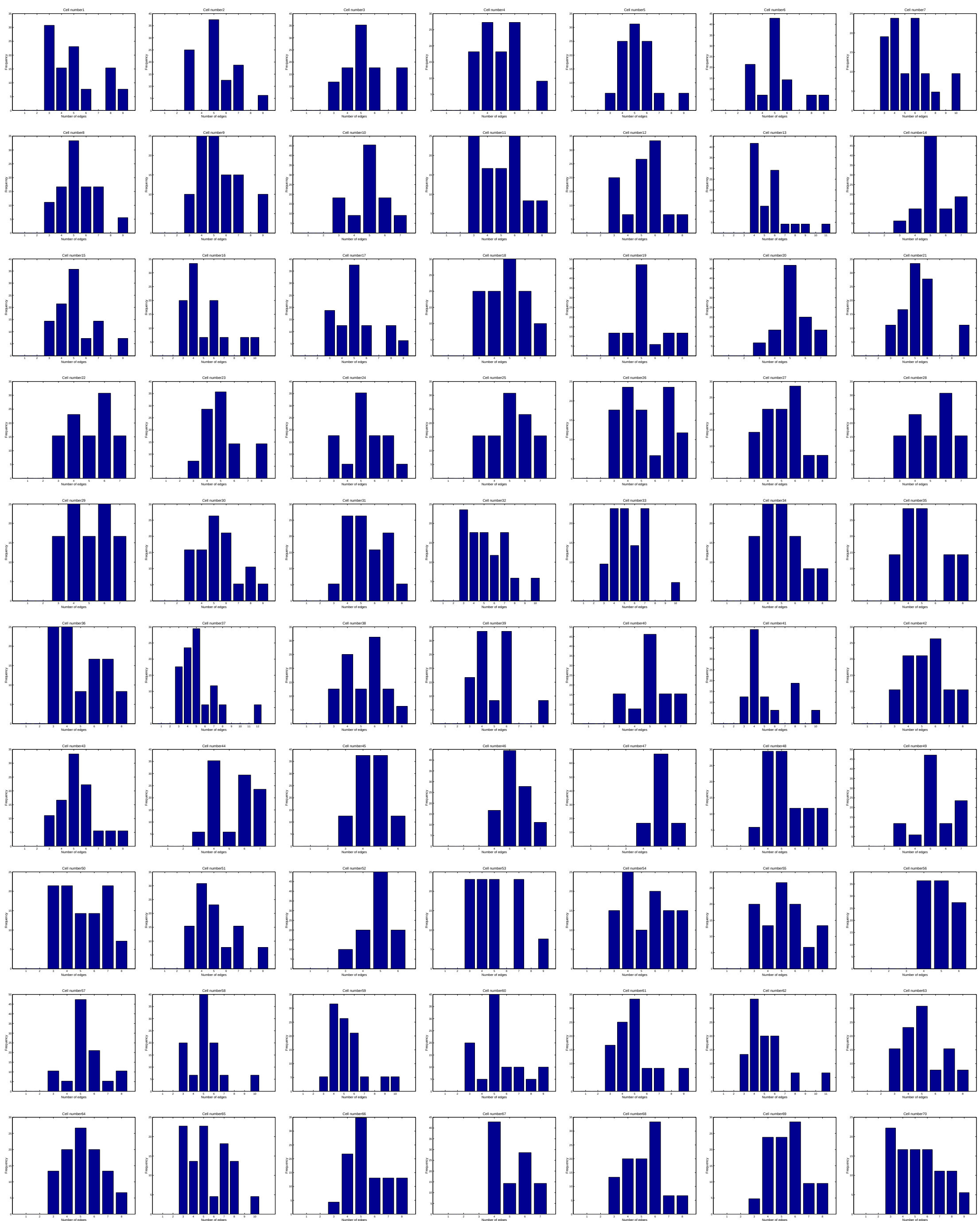


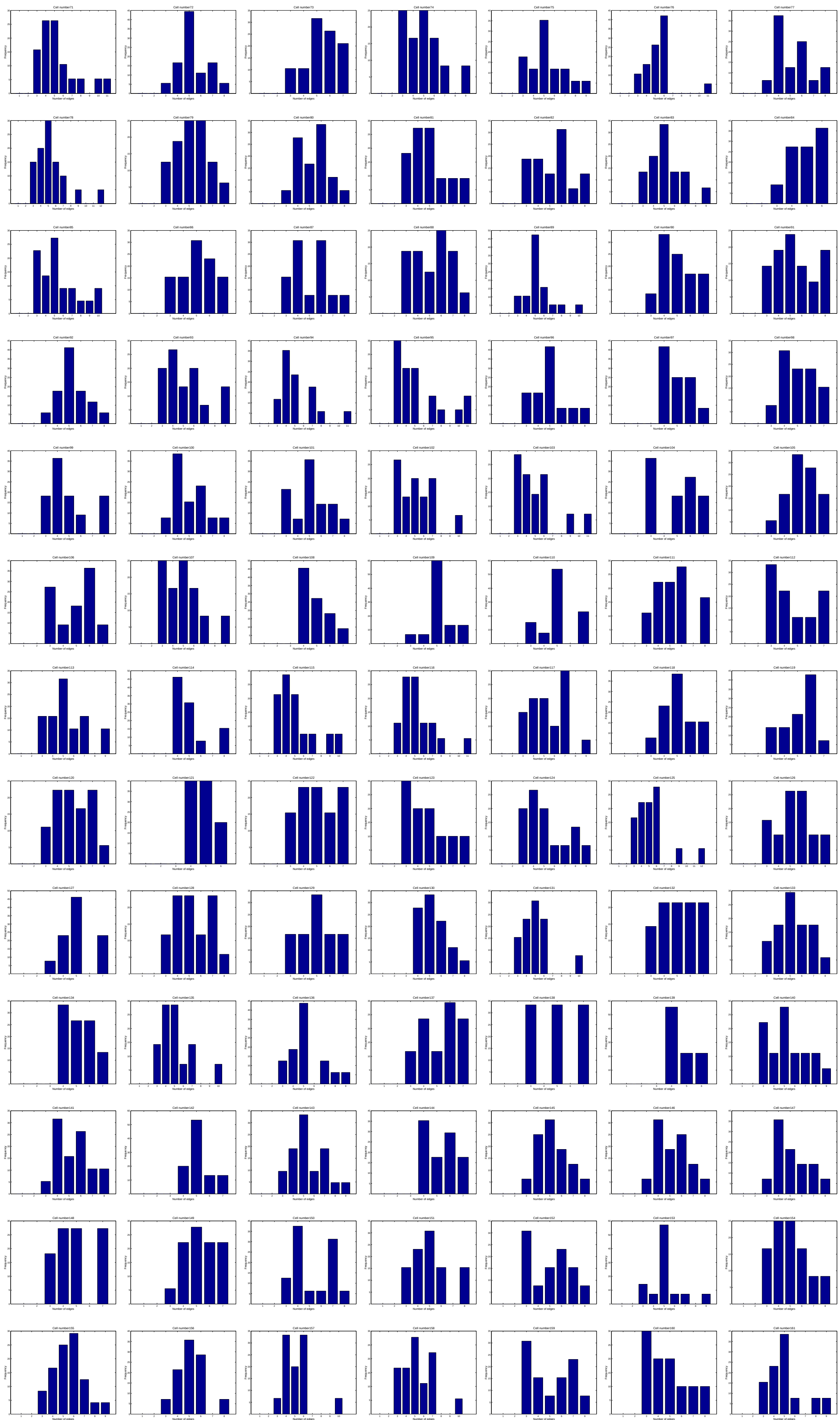
The characteristic area is the area of each square in the assembly the total volume and the number of cells of which are the same as those of the Voronoi graph. The reference curve shown in Figure 2.7 is $y = \frac{0.43}{x^{0.9}}$.

Figure 2.7 Characteristic area versus the number of cells.

	n_c^v	n_f^v	$\mathcal{N}_v^c(\mathcal{P}_c^f)$	$\mathcal{N}_v^c(A_f)$	$\mathcal{N}_v^c(A_c)$	$\mathcal{N}_v^c(V_c)$	n_c^f
min	8	3	0.001414	4.6179×10^{-7}	0.11908	0.21464	6
max	42	10	0.84885	0.82919	0.50425	12.719	23
μ	24.423	5.1114	0.32876	0.12535	0.30315	1.0000	14.211
σ^2	44.105	2.1650	0.030844	0.015846	0.008511	2.5764	11.026
μ_g	23.504	4.9065	0.25897	0.051216	0.28815	0.70118	13.825
μ_h	22.506	4.7086	0.12620	2.50×10^{-4}	0.27198	0.58794	13.423
med	22	5	0.33219	0.084165	0.30368	0.64077	13
mad	5.4338	1.1655	0.14499	0.099433	0.075445	0.67950	2.7169
$\mathcal{M}^2$	43.483	2.1619	0.030800	0.015823	0.008391	2.5401	10.871
$\mathcal{M}^3$	112.13	1.8902	5.92×10^{-4}	0.002805	4.8×10^{-5}	24.169	14.016
$\mathcal{K}$	3.1096	3.0234	2.3090	5.2926	2.3015	42.238	3.1096

Table 2.2 *From rbox (200 random points, seed 34565473) and qhull (option v and o);* $d = 3$, $n^c = 71$.





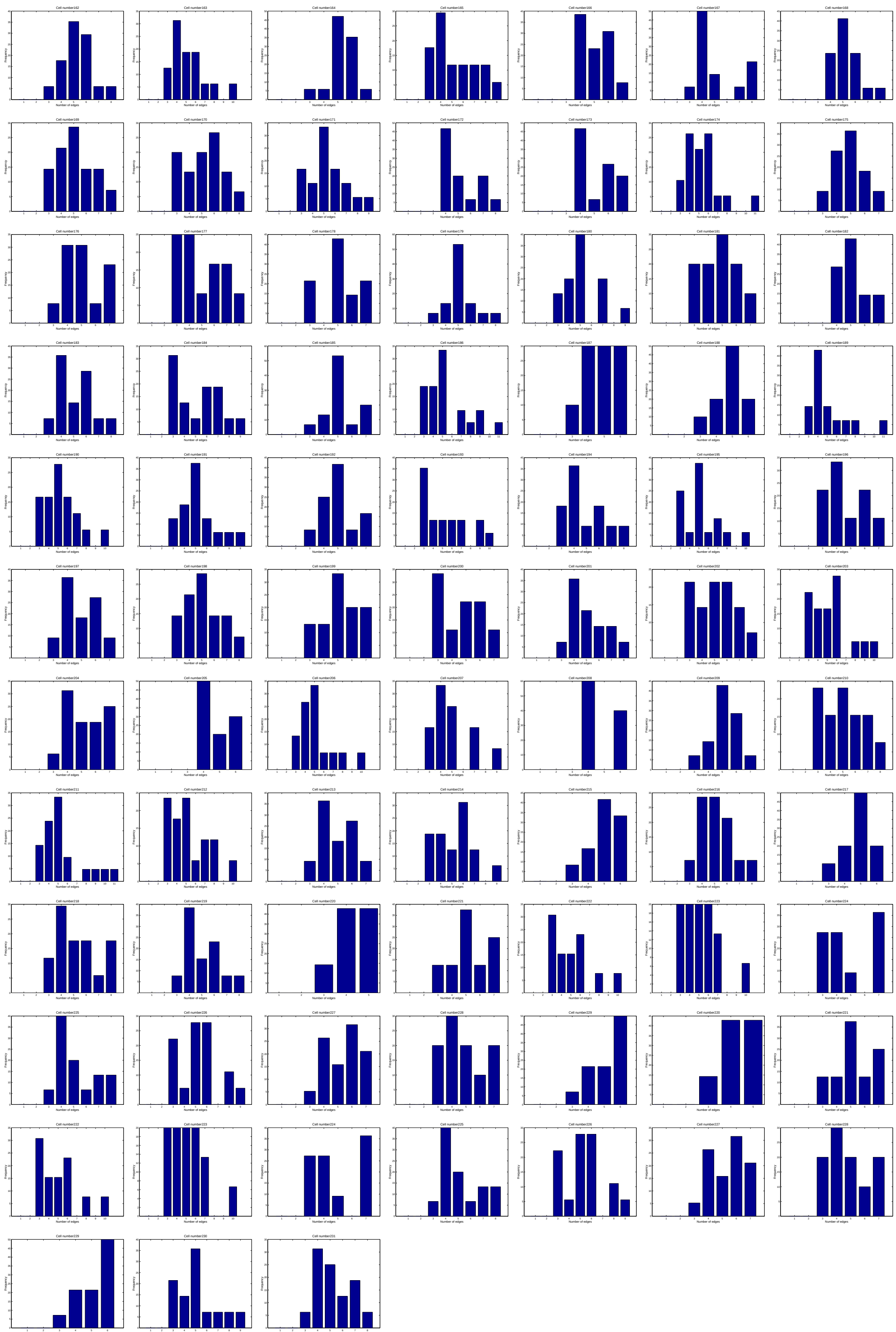
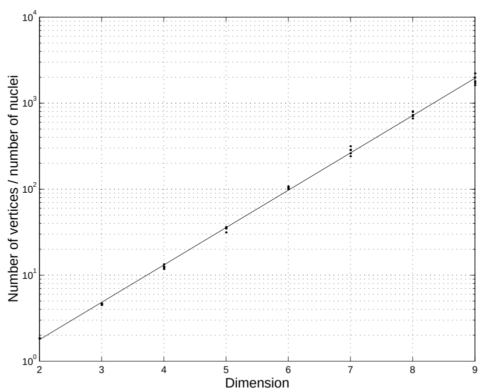


Figure 2.8 *Distribution of the number of edges per face. Each picture is an individual cell. The distribution shows the relative abundance or the number of faces (the vertical axes) having the number of edges as shown by the horizontal axes. The horizontal axis scales are positive integers starting from zero at the origin. Simulation uses rbox (500 random points, seed 893280) and qhull (option v and o); $d = 3$, $n^c = 231$, $n^v = 3107$, CPU time 81.1 sec for finding face area, 2.22 sec for cell volume, 49.59 sec for counting edges and 0.03 sec for finding the number of edges and faces.*

The minimum number of edges for each face is three. This number is the same as the number of vertices of that face. From these figures most of the cells have at least one face with three edges. There are only 19 cells (8.23 percents) which does not have any three-edged face, and all of them have some four-edged faces. Therefore there is no cell with five as the minimum number of edges per face. The maximum number of edges per face is less clear-cut. There are twelve cells (5.19 percents) with 11 as the maximum number of edges per face and two (0.87 percents) with 12.

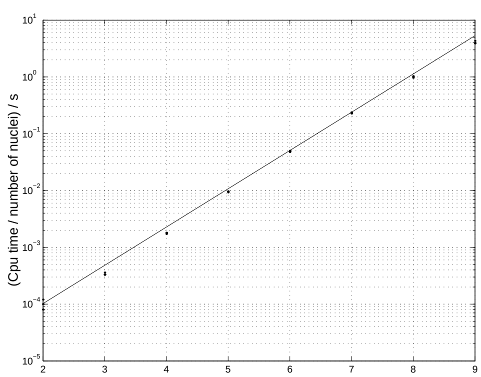
	n_c^v	$\aleph_c$	A_c^{fr}	V_c^{fr}	α	n_c^e	$E(n^e_{f,c})$	$\sigma(n^e_{f,c})$
min	10	7	1.0989×10^{-3}	2.5790×10^{-4}	0.0406	15	4.2857	0.6030
max	44	24	1.0956×10^{-2}	0.24840	4.2505	66	5.5000	2.7028
μ	25.974	14.987	4.3290×10^{-3}	4.3290×10^{-3}	1.2515	38.961	5.1601	1.5450
σ^2	40.852	10.213	3.3308×10^{-6}	2.9419×10^{-4}	0.3802	91.916	0.0372	0.1243
σ	6.3915	3.1958	1.8250×10^{-3}	1.7152×10^{-2}	0.6166	9.5873	0.1928	0.3526
μ_g	25.166	14.642	3.9633×10^{-3}	1.8465×10^{-3}	1.0816	37.749	5.1564	1.5030
μ_h	24.323	14.288	3.6026×10^{-3}	1.2471×10^{-3}	0.8241	36.484	5.1526	1.4580
med	26	15	4.1467×10^{-3}	1.5715×10^{-3}	1.1915	39	5.2000	1.5315
mad	5.1532	2.5766	1.4064×10^{-3}	4.5864×10^{-3}	0.4720	7.7298	0.1505	0.2794
$\mathcal{M}^2$	40.675	10.169	3.3163×10^{-6}	2.9291×10^{-4}	0.3785	91.518	0.0370	0.1238
$\mathcal{M}^3$	59.377	7.4222	5.7258×10^{-9}	6.3817×10^{-5}	0.2143	2.0040×10^{2}	-0.0071	0.0058
$\mathcal{M}^4$	4.5398×10^{3}	2.8374×10^{2}	4.6684×10^{-11}	1.5395×10^{-5}	0.7390	2.2983×10^{4}	0.0063	0.0463
$\mathcal{K}$	2.7440	2.7440	4.2448	1.7943×10^{2}	5.1578	2.7440	4.5730	3.0230

Because the Voronoi tessellation being studied is simple, $\aleph_c^v = \aleph_c^f = n_c^f$. In other words, any two cells having at least one vertex in common are neighbours to each other, and the number of neighbours around any cell in an infinite network is equal to the number of its faces.



The line shown in Figure 2.9 is $y = 0.2410e^n$, where n is the dimension of the network and is the horizontal axis; the coefficient of the exponential term is obtained by averaging over the averages in each dimension, each of which in turn comes from five batch runs.

Figure 2.9 Ratios between the number of vertices and the number of cells in Voronoi networks of various dimensions.



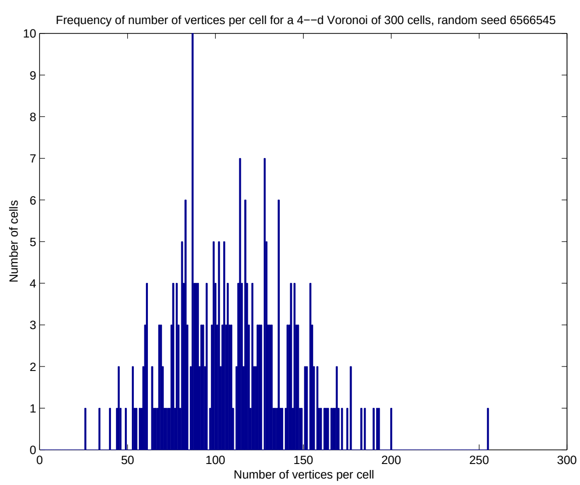
The line shown in Figure 2.10, found manually by trial and error, has the equation $y = 4.61 \times 10^{-6}(2 + e)^n$. In comparison, substituting the average cpu time for each dimension for y in the equation $y = Ae^n$ to obtain A and then find the average again over all dimensions results in $\bar{A} = 1.612 \times 10^{-4}$. The program which produces both Figure 2.10 and $\aleph$ is given in § 3.8.

Figure 2.10 Cpu time in creating the Voronoi networks for Figure 2.10.

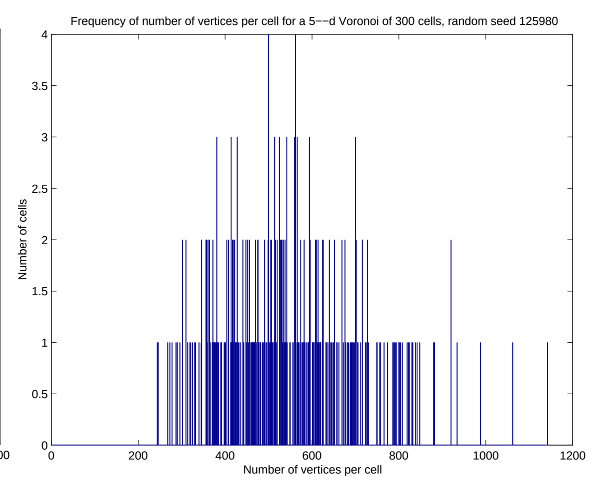
The number of vertices of higher dimensions is investigated briefly in the following Table 2.3 and Figure 2.11 the simulation of which was carried out by the program in § $\aleph$.

Dimension	4	5	6	8	9	10
N_c	300	300	300	30	30	30
N_v	6,577	27,150	118,534	5,465	10,467	17,442
n_c	132	104	14	2	3	0
$\min(n_c^v)$	26	244.00	1,020.0	952.002	2,353.0	4,287.0
$\max(n_c^v)$	255	1,142.0	5,852.0	2,638.0	5,534.0	9,528.0
$\bar{n}_c^v$	109.90	543.51	2,766.5	1,640.2	3,489.7	6,396.0
$(\sigma_{n_v}^2)_c$	1,155.9	22,974	6.0898×10^5	1.7964×10^5	7.0328×10^5	2.3635×10^6
$\sigma_c^{n_v}$	33.999	151.57	780.37	423.84	838.62	1.5374×10^3
$\mu_g(n_c^v)$	104.41	523.06	2,666.1	1,589.9	3,398.4	6,225.6
$\mu_h(n_c^v)$	98.389	502.94	2,571.4	1,542.3	3,313.9	6,065.2
$\mathrm{med}(n_c^v)$	107.50	527.00	2,651.0	1,568.5	3,272.5	5,816.5
$\mathrm{mad}(n_c^v)$	27.366	119.39	588.19	331.96	687.47	1,296.6
$m^2(n_c^v)$	1,152.1	22,897	6.0695×10^5	1.7365×10^5	6.7984×10^5	2.2847×10^6
$m^3(n_c^v)$	15,555	2.2871×10^6	5.1556×10^8	4.7965×10^7	3.9108×10^8	1.8988×10^9
$m^4(n_c^v)$	4.5760×10^6	1.8905×10^9	1.7964×10^{12}	8.5043×10^{10}	1.1967×10^{12}	1.1638×10^{13}
$\kappa(n_c^v)$	3.4476	3.6059	4.8764	2.8202	2.5892	2.2296

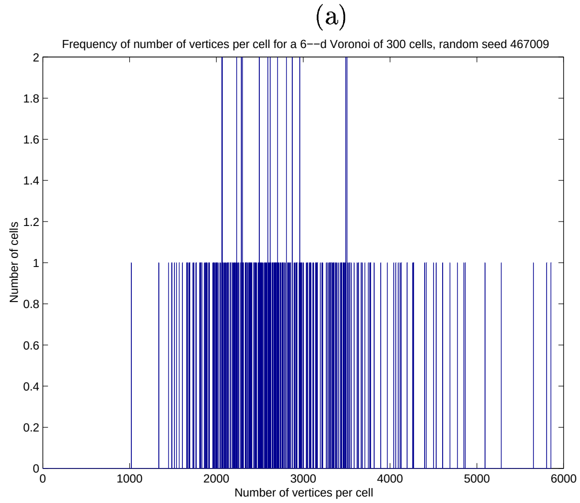
Table 2.3 *Statistics of the number of vertices in 4, 5, 6, 8, 9, and 10 dimensions*



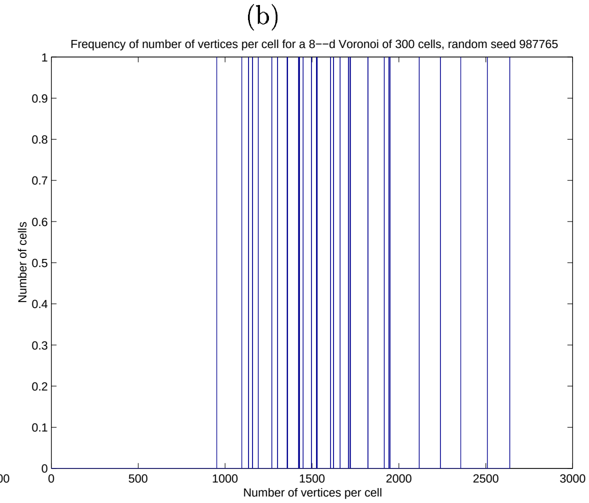
(a)



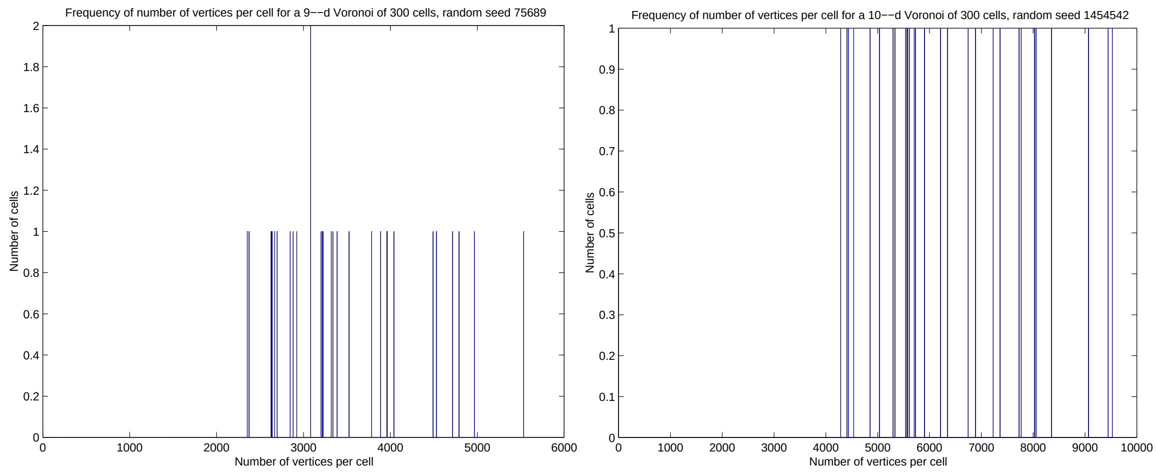
(b)



(c)



(d)

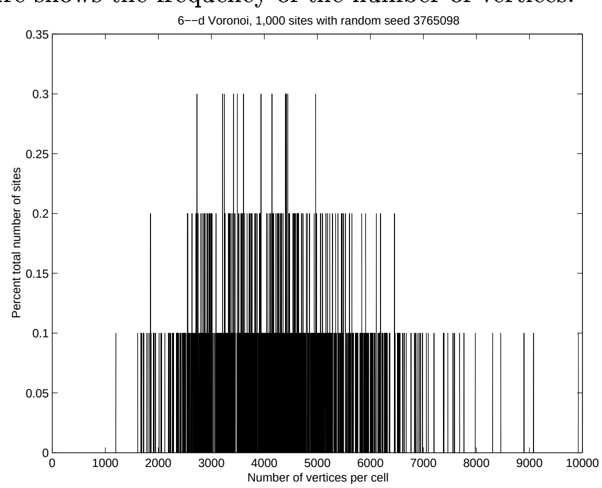


(e) (f)

Figure 2.11 *Distribution of vertices in* (a) *4,* (b) *5,* (c) *6, and* (d) *8,* (e) *9,* (f) *10 dimensions*

To obtain the number of vertices per cell, n_c^v, of a six-dimensional Voronoi structure of 1,000 cells I used a batch program, for instance the one listed in § ℵ. The Matlab macro that this program refers to opens and reads from a file the number of vertices and then finds the statistical values, *viz.* $\min(n_c^v) = 1,198$; $\max(n_c^v) = 9923$; $\bar{n}_c^v = 4201.1$; $(\sigma^2_{n_v})_c = 1.4069 \times 10^6$; $\sigma_c^{n_v} = 1186.1$; $(\bar{n}_g)_c^v = 4035.4$; $(\bar{n}_h)_c^v = 3866.8$; $\mathrm{med}(n_c^v) = 4122.5$; $\mathrm{mad}(n_c^v) = 931.81$; $m^2(n_c^v) = 1.4055 \times 10^6$; $m^3(n_c^v) = 1.0462 \times 10^9$; $m^4(n_c^v) = 7.7148 \times 10^{12}$; and $\kappa(n_c^v) = 3.9053$.

The following figure shows the frequency of the number of vertices.



There are three cells with 2729, 3213, 3246, 3421, 3490, 3606, 3938, 4143, 4398, 4417, 4442 and 4970 vertices. There are two having 1854, 2549, 2557, 2634, 2712, 2720, 2722, 2751, 2804, 2843, 2869, 2878, 2882, 2920, 2931, 2957, 2973, 2996, 3013, 3090, 3260, 3322, 3329, 3343, 3366, 3393, 3418, 3424, 3446, 3513, 3520, 3560, 3577, 3583, 3585, 3610, 3631, 3677, 3714, 3732, 3753, 3767, 3770, 3819, 3835, 3861, 3907, 3919, 3924, 3937, 3939, 4045, 4053, 4091, 4117, 4122, 4123, 4163, 4178, 4219, 4251, 4261, 4269, 4272, 4298, 4317, 4327, 4355, 4461, 4470, 4473, 4474, 4542, 4563, 4572, 4576, 4586, 4618, 4624, 4629, 4640, 4646, 4648, 4700, 4729, 4792, 4810, 4851, 4942, 4975, 4984, 4985, 5058, 5064, 5097, 5161, 5195, 5235, 5287, 5347, 5387, 5455, 5467, 5492, 5529, 5606, 5650, 5838, 5913, 6112, 6193 and 6455 vertices. And there is exactly one cell with each of the following numbers of vertices, 1198, 1612, 1672, 1686, 1688, 1717, 1727, 1787, 1827, 1864, 1906, 1915, 1921, 1947, 2016, 2052, 2056, 2060, 2123, 2190, 2200, 2223, 2231, 2236, 2267, 2280, 2281, 2286, 2289, 2344, 2353, 2356, 2371, 2372, 2385, 2408, 2423, 2426, 2429, 2455, 2475, 2487, 2499, 2500, 2503, 2504, 2508, 2513, 2542, 2547, 2551, 2560, 2561, 2565, 2575, 2578, 2580, 2585, 2586, 2596, 2617, 2619, 2622, 2650, 2654, 2658, 2664, 2669, 2673, 2686, 2692, 2694, 2704, 2708, 2716, 2718, 2723, 2732, 2733, 2742, 2743, 2755, 2771, 2779, 2781, 2783, 2791, 2800, 2807, 2808, 2811, 2820, 2835, 2837, 2853, 2858, 2864, 2867, 2870, 2873, 2875, 2880, 2884, 2887, 2893, 2894, 2895, 2902, 2907, 2910, 2914, 2916, 2925, 2928, 2934, 2949, 2955, 2956, 2960, 2967, 2974, 2978, 2983, 2985, 2989, 2993, 3007, 3012, 3014, 3027, 3051, 3074, 3102, 3107, 3114, 3116, 3119, 3129, 3130, 3136, 3137, 3141, 3148, 3152, 3164, 3173, 3176, 3180, 3182, 3186, 3188, 3190, 3192, 3199, 3202, 3204, 3206, 3208, 3219,

3221, 3229, 3231, 3237, 3239, 3244, 3245, 3255, 3256, 3259, 3267, 3270, 3271, 3274, 3275, 3285, 3295, 3296, 3305, 3307, 3312, 3316, 3317, 3318, 3319, 3333, 3337, 3344, 3351, 3352, 3354, 3357, 3360, 3370, 3373, 3374, 3378, 3384, 3390, 3394, 3396, 3402, 3403, 3408, 3427, 3428, 3436, 3440, 3443, 3444, 3452, 3453, 3454, 3486, 3487, 3492, 3499, 3502, 3505, 3506, 3509, 3519, 3521, 3522, 3523, 3536, 3541, 3544, 3545, 3547, 3549, 3551, 3555, 3556, 3573, 3575, 3578, 3581, 3582, 3587, 3589, 3594, 3595, 3599, 3604, 3609, 3611, 3620, 3628, 3634, 3635, 3636, 3639, 3647, 3652, 3656, 3658, 3663, 3666, 3668, 3680, 3685, 3686, 3687, 3689, 3691, 3694, 3695, 3696, 3706, 3709, 3710, 3713, 3716, 3717, 3727, 3728, 3735, 3739, 3740, 3742, 3743, 3744, 3751, 3764, 3772, 3775, 3776, 3778, 3782, 3787, 3789, 3792, 3794, 3798, 3803, 3804, 3806, 3808, 3810, 3815, 3818, 3821, 3830, 3834, 3837, 3838, 3842, 3847, 3849, 3850, 3863, 3867, 3868, 3883, 3884, 3889, 3890, 3899, 3900, 3902, 3903, 3908, 3912, 3914, 3918, 3920, 3925, 3929, 3942, 3943, 3944, 3947, 3949, 3956, 3957, 3960, 3962, 3963, 3978, 3980, 3981, 3986, 3988, 3996, 4008, 4016, 4017, 4019, 4027, 4030, 4033, 4035, 4038, 4039, 4044, 4049, 4057, 4072, 4073, 4075, 4077, 4082, 4086, 4092, 4106, 4111, 4112, 4124, 4126, 4128, 4134, 4140, 4145, 4146, 4147, 4148, 4150, 4153, 4157, 4162, 4176, 4177, 4179, 4188, 4189, 4190, 4193, 4195, 4203, 4206, 4212, 4214, 4217, 4218, 4225, 4227, 4232, 4238, 4239, 4244, 4245, 4246, 4247, 4257, 4262, 4271, 4273, 4279, 4283, 4288, 4291, 4293, 4300, 4308, 4313, 4316, 4320, 4322, 4324, 4326, 4331, 4334, 4335, 4337, 4339, 4342, 4346, 4349, 4350, 4351, 4352, 4354, 4357, 4361, 4376, 4377, 4386, 4388, 4395, 4399, 4405, 4410, 4411, 4413, 4425, 4426, 4428, 4447, 4449, 4451, 4453, 4459, 4471, 4475, 4480, 4488, 4493, 4496, 4501, 4504, 4506, 4512, 4515, 4518, 4522, 4523, 4532, 4548, 4552, 4556, 4561, 4574, 4580, 4585, 4587, 4595, 4598, 4602, 4605, 4610, 4614, 4619, 4621, 4626, 4628, 4632, 4634, 4639, 4649, 4651, 4657, 4663, 4665, 4674, 4681, 4684, 4697, 4702, 4703, 4705, 4710, 4711, 4714, 4725, 4728, 4736, 4738, 4740, 4751, 4754, 4755, 4761, 4765, 4770, 4779, 4781, 4814, 4816, 4826, 4829, 4831, 4844, 4850, 4873, 4877, 4880, 4885, 4890, 4892, 4896, 4899, 4900, 4902, 4903, 4905, 4908, 4911, 4914, 4915, 4919, 4924, 4933, 4937, 4944, 4955, 4960, 4967, 4969, 4974, 4992, 4994, 4997, 5000, 5002, 5003, 5007, 5019, 5021, 5028, 5029, 5035, 5039, 5050, 5074, 5080, 5083, 5092, 5093, 5104, 5108, 5110, 5117, 5119, 5120, 5131, 5138, 5156, 5158, 5166, 5167, 5176, 5185, 5192, 5194, 5200, 5203, 5219, 5225, 5226, 5233, 5236, 5239, 5251, 5259, 5265, 5270, 5271, 5274, 5285, 5288, 5290, 5313, 5316, 5318, 5325, 5328, 5336, 5337, 5346, 5364, 5389, 5405, 5408, 5415, 5432, 5439, 5440, 5448, 5457, 5460, 5463, 5464, 5466, 5468, 5470, 5474, 5482, 5484, 5530, 5549, 5567, 5589, 5591, 5598, 5612, 5627, 5632, 5664, 5676, 5680, 5718, 5719, 5720, 5724, 5738, 5744, 5747, 5748, 5749, 5752, 5764, 5802, 5806, 5824, 5841, 5859, 5880, 5885, 5892, 5896, 5899, 5927, 5928, 5947, 5956, 5957, 5966, 5998, 6006, 6014, 6017, 6031, 6035, 6038, 6040, 6054, 6055, 6075, 6081, 6084, 6089, 6092, 6113, 6145, 6166, 6169, 6180, 6192, 6195, 6221, 6237, 6265, 6272, 6273, 6286, 6303, 6305, 6310, 6313, 6320, 6356, 6367, 6456, 6469, 6510, 6515, 6520, 6521, 6524, 6541, 6561, 6609, 6615, 6642, 6677, 6767, 6771, 6831, 6852, 6883, 6918, 6945, 6985, 7056, 7093, 7201, 7379, 7387, 7461, 7560, 7588, 7685, 7764, 7765, 7979, 8308, 8460, 8899, 9079 and 9923. But these results are not very useful since bordering cells were not excluded.

Result on 700 cells

```
Number of cells: 700
Box size: 10
No compression
Number of cells: 700
Number of vertices: 4380
Number of cells in frame: 341
Time for counting stats: 2191.39 seconds
Number of faces connected to the first vertice at infinity: 181
Time for finding cell volumes: 3.150000
```

n_c 700	n_v 4,380	$n_{f,1^{st}v}$ 181	$n_{c_{in}}$ 341	$t_{CPU,stat.}$ 2191.39 sec.	$t_{CPU,A}$ 137.13 sec.	$t_{CPU,V}$ 3.15 sec.	
$\min n_{v_c}$ 10	$\max n_{v_c}$ 46	$\bar{n_{v_c}}$ 25.106	$\sigma^2_{n_{v_c}}$ 46.784	$\sigma_{n_{v_c}}$ 6.8399	$g, \bar{n}_{v_c}$ 24.169	$h, \bar{n}_{v_c}$ 23.208	n_{v_c}
$\widetilde{n_{v_c}}$ 24	$\delta_\mu(n_{v_c})$ 5.4023	$M^2(n_{v_c})$ 46.717	$M^3(n_{v_c})$ 1.4240×10^2	$M^4(n_{v_c})$ 6.5877×10^3	$\kappa_{n_v,c}$ 3.0184		
$\min n_{v,c_{in}}$ 10	$\max n_{v,c_{in}}$ 46	$\bar{v_{c_{in}}}$ 26.246	$\sigma^2_{v,c_{in}}$ 40.974	$\sigma_{v,c_{in}}$ 6.4011	$\bar{v}_{,gc_{in}}$ 25.464	$\bar{v}_{,hc_{in}}$ 24.668	$n_{v,c_{in}}$
$\widetilde{n}_{v,c_{in}}$ 26	$\delta_\mu(n_{v,c_{in}})$ 5.1101	$M^2(n_{v,c_{in}})$ 40.854	$M^3(n_{v,c_{in}})$ 1.0338×10^2	$M^4(n_{v,c_{in}})$ 4.9173×10^3	$\kappa_{n_{v,c_{in}}}$ 2.9461		
$\min n_{\aleph_v,c_{in}}$ 8	$\max n_{\aleph_v,c_{in}}$ 26	$\bar{n}_{\aleph_v,c_{in}}$ 16.123	$\sigma^2_{n_{\aleph_v,c_{in}}}$ 10.244	$\sigma_{n_{\aleph_v,c_{in}}}$ 3.2006	$\bar{n}_{g,\aleph_v,c_{in}}$ 15.809	$\bar{n}_{h,\aleph_v,c_{in}}$ 15.495	$n_{\aleph_v,c_{in}}$
$\widetilde{n}_{\aleph_v,c_{in}}$ 16	$\delta_\mu(n_{\aleph_v,c_{in}})$ 2.5550	$M^2(n_{\aleph_v,c_{in}})$ 10.214	$M^3(n_{\aleph_v,c_{in}})$ 12.922	$M^4(n_{\aleph_v,c_{in}})$ 3.0733×10^2	$\kappa_{n_{\aleph_v,c_{in}}}$ 2.9461		
$\min n_{\aleph_e,c}$ 8	$\max n_{\aleph_e,c}$ 27	$\bar{n}_{\aleph_e,c}$ 15.669	$\sigma^2_{n,\aleph_e,c}$ 12.439	$\sigma_{n,\aleph_e,c}$ 3.5269	$\bar{n}_{g,\aleph_e,c}$ 15.279	$\bar{n}_{h,\aleph_e,c}$ 14.894	$n_{\aleph_e,c}$
$\widetilde{n}_{\aleph_e,c}$ 15	$\delta_\mu(n_{\aleph_e,c})$ 2.7792	$M^2(n_{\aleph_e,c})$ 12.422	$M^3(n_{\aleph_e,c})$ 22.761	$M^4(n_{\aleph_e,c})$ 4.8871×10^2	$\kappa_{n,\aleph_e,c}$ 3.1673		
$\min n_{\aleph_e,c_{in}}$ 8	$\max n_{\aleph_e,c_{in}}$ 26	$\bar{n}_{\aleph_e,c_{in}}$ 16.123	$\sigma^2_{n(\aleph_e),c_{in}}$ 10.244	$\sigma_{n(\aleph_e),c_{in}}$ 3.2006	$\bar{n}_{g,\aleph_e,c_{in}}$ 15.809	$\bar{n}_{h,\aleph_e,c_{in}}$ 15.495	$n_{\aleph_e,c_{in}}$
$\widetilde{n}_{\aleph_e,c_{in}}$ 16	$\delta_\mu(n_{\aleph_e,c_{in}})$ 2.5550	$M^2(n_{\aleph_e,c_{in}})$ 10.214	$M^3(n_{\aleph_e,c_{in}})$ 12.922	$M^4(n_{\aleph_e,c_{in}})$ 3.0733×10^2	$\kappa_{n(\aleph_e),c_{in}}$ 2.9461		
$\min n_{\aleph_f,c}$ 8	$\max n_{\aleph_f,c}$ 27	$\bar{n}_{\aleph_f,c}$ 15.629	$\sigma^2_{n(\aleph_f),c}$ 12.274	$\sigma_{n(\aleph_f),c}$ 3.5034	$\bar{n}_{g,\aleph_f,c}$ 15.242	$\bar{n}_{h,\aleph_f,c}$ 14.857	$n_{\aleph_f,c}$
$\widetilde{n}_{\aleph_f,c}$ 15	$\delta_\mu(n_{\aleph_f,c})$ 2.7635	$M^2(n_{\aleph_f,c})$ 12.256	$M^3(n_{\aleph_f,c})$ 21.051	$M^4(n_{\aleph_f,c})$ 4.6791×10^2	$\kappa_{n(\aleph_f),c}$ 3.1149		
$\min n_{\aleph_f,c_{in}}$ 8	$\max n_{\aleph_f,c_{in}}$ 26	$\bar{n}_{\aleph_f,c_{in}}$ 16.123	$\sigma^2_{n(\aleph_f),c_{in}}$ 10.244	$\sigma_{n(\aleph_f),c_{in}}$ 3.2006	$\bar{n}_{g,\aleph_f,c_{in}}$ 15.809	$\bar{n}_{h,\aleph_f,c_{in}}$ 15.495	$n_{\aleph_f,c_{in}}$
$\widetilde{n}_{\aleph_f,c_{in}}$ 16	$\delta_\mu(n_{\aleph_f,c_{in}})$ 2.5550	$M^2(n_{\aleph_f,c_{in}})$ 10.214	$M^3(n_{\aleph_f,c_{in}})$ 12.922	$M^4(n_{\aleph_f,c_{in}})$ 3.0733×10^2	$\kappa_{n(\aleph_f),c_{in}}$ 2.9461		
$\min A_c$ 0.59209	$\max A_c$ 83.165	$\bar{A}_c$ 3.4086	$\sigma^2_{A_c}$ 2.0373	σ_{A_c} 2.3046×10^2	$\bar{A}g, c$ 5.0726	$\bar{A}h, c$ 1.0502	A_c
$\widetilde{A}_c$ 1.1935×10^2	$\delta_\mu(A_c)$ 4.0384	$M^2(A_c)$ 2.8300	$M^3(A_c)$ 2.0530	$M^4(A_c)$ 1.3800	κ_{A_c} 2.0572		
$\min A_{c_{in}}$ 0.59209	$\max A_{c_{in}}$ 3.4086	$\bar{A}_{c_{in}}$ 2.0373	$\sigma^2_{A_{c_{in}}}$ 1.0502	$\sigma_{A_{c_{in}}}$ 4.0384	$\bar{A}g, c_{in}$ 2.8300	$\bar{A}h, c_{in}$ 2.0530	$A_{c_{in}}$
$\widetilde{A}_{c_{in}}$ 1.3800	$\delta_\mu(A_{c_{in}})$ 2.0572	$M^2(A_{c_{in}})$ 2.9782	$M^3(A_{c_{in}})$ 3.9125	$M^4(A_{c_{in}})$ 3.1031	$\kappa_{A_{c_{in}}}$ 1.0800		
$\min A_{fr,c}$ 8.5015×10^{-8}	$\max A_{fr,c}$ 1.1941×10^{-5}	$\bar{A}_{fr,c}$ 4.8942×10^{-7}	$\sigma^2_{A_{fr,c}}$ 2.9253×10^{-7}	$\sigma_{A_{fr,c}}$ 3.3091×10^{-5}	$\bar{A}g, fr, c$ 7.2835×10^{-7}	$\bar{A}h, fr, c$ 1.5079×10^{-7}	$A_{fr,c}$
$\widetilde{A}_{fr,c}$ 1.7137×10^{-5}	$\delta_\mu(A_{fr,c})$ 5.7986×10^{-7}	$M^2(A_{fr,c})$ 4.0635×10^{-7}	$M^3(A_{fr,c})$ 2.9478×10^{-7}	$M^4(A_{fr,c})$ 1.9816×10^{-7}	$\kappa_{A_{fr,c}}$ 2.9538×10^{-7}		
$\min A_{fr,c_{in}}$ 7.5700×10^{-4}	$\max A_{fr,c_{in}}$ 4.3579×10^{-3}	$\bar{A}_{fr,c_{in}}$ 2.6048×10^{-3}	$\sigma^2_{A_{fr,c_{in}}}$ 1.3427×10^{-3}	$\sigma_{A_{fr,c_{in}}}$ 5.1632×10^{-3}	$\bar{A}g, fr, c_{in}$ 3.6182×10^{-3}	$\bar{A}h, fr, c_{in}$ 2.6248×10^{-3}	$A_{fr,c_{in}}$
$\widetilde{A}_{fr,c_{in}}$ 1.7644×10^{-3}	$\delta_\mu(A_{fr,c_{in}})$ 2.6301×10^{-3}	$M^2(A_{fr,c_{in}})$ 3.8077×10^{-3}	$M^3(A_{fr,c_{in}})$ 5.0023×10^{-3}	$M^4(A_{fr,c_{in}})$ 3.9674×10^{-3}	$\kappa_{A_{fr,c_{in}}}$ 1.3808×10^{-3}		
$\min V_c$ 0.88511	$\max V_c$ 6.2052×10^6	$\bar{V}_c$ 2.7455×10^4	$\sigma^2_{V_c}$ 8.9303×10^{10}	σ_{V_c} 2.9884×10^5	$\bar{V}g, c$ 22.548	$\bar{V}h, c$ 6.3296	V_c
$\widetilde{V}_c$ 7.6212	$\delta_\mu(V_c)$ 5.1126×10^4	$M^2(V_c)$ 8.9176×10^{10}	$M^3(V_c)$ 4.3202×10^{17}	$M^4(V_c)$ 2.3781×10^{24}	κ_{V_c} 2.9904×10^2		
$\min V_{c_{in}}$ 0.88511	$\max V_{c_{in}}$ 1.8958×10^3	$\bar{V}_{c_{in}}$ 20.542	$\sigma^2_{V_{c_{in}}}$ 1.5264×10^4	$\sigma_{V_{c_{in}}}$ 1.2355×10^2	$\bar{V}g, c_{in}$ 5.9195	$\bar{V}h, c_{in}$ 4.4392	$V_{c_{in}}$
$\widetilde{V}_{c_{in}}$ 5.4355	$\delta_\mu(V_{c_{in}})$ 27.858	$M^2(V_{c_{in}})$ 1.5219×10^4	$M^3(V_{c_{in}})$ 2.2911×10^7	$M^4(V_{c_{in}})$ 3.9178×10^{10}	$\kappa_{V_{c_{in}}}$ 1.6915×10^2		
$\min V_{fr,c}$ 4.6056×10^{-8}	$\max V_{fr,c}$ 0.32288	$\bar{V}_{fr,c}$ 1.4286×10^{-3}	$\sigma^2_{V_{fr,c}}$ 2.4179×10^{-4}	$\sigma_{V_{fr,c}}$ 1.5550×10^{-2}	$\bar{V}g, fr, c$ 1.1733×10^{-6}	$\bar{V}h, fr, c$ 3.2935×10^{-7}	$V_{fr,c}$
$\widetilde{V}_{fr,c}$ 3.9656×10^{-7}	$\delta_\mu(V_{fr,c})$ 2.6603×10^{-3}	$M^2(V_{fr,c})$ 2.4145×10^{-4}	$M^3(V_{fr,c})$ 6.0865×10^{-5}	$M^4(V_{fr,c})$ 1.7433×10^{-5}	$\kappa_{V_{fr,c}}$ 2.9904×10^2		
$\min V_{fr,c_{in}}$ 1.2636×10^{-4}	$\max V_{fr,c_{in}}$ 0.27064	$\bar{V}_{fr,c_{in}}$ 2.9326×10^{-3}	$\sigma^2_{V_{fr,c_{in}}}$ 3.1108×10^{-4}	$\sigma_{V_{fr,c_{in}}}$ 1.7638×10^{-2}	$\bar{V}g, fr, c_{in}$ 8.4507×10^{-4}	$\bar{V}h, fr, c_{in}$ 6.3374×10^{-4}	$V_{fr,c_{in}}$
$\widetilde{V}_{fr,c_{in}}$ 7.7598×10^{-4}	$\delta_\mu(V_{fr,c_{in}})$ 3.9770×10^{-3}	$M^2(V_{fr,c_{in}})$ 3.1017×10^{-4}	$M^3(V_{fr,c_{in}})$ 6.6659×10^{-5}	$M^4(V_{fr,c_{in}})$ 1.6273×10^{-5}	$\kappa_{V_{fr,c_{in}}}$ 1.6915×10^2		

AVS

The first picture I created on AVS was a Trigonal Dipyramidal. The program was the following .inp file.

```
5 2 0 0 0
1 0 0 1.2910
2 2 0 1.2910
3 1 1.7321 1.2910
4 1 0.5774 0
5 1 0.5774 2.5820
1 1 tet 1 2 3 4
2 1 tet 1 2 3 5
1 2
```

The connection of modules was ReadUCD to ExternalEdges to UViewer3D.

Problems related to ISD and NQS

Some of the staffs at our Information Systems Department are unprofessional in their jobs. Nicholas Blackaby, for example, has once wiped out all about seven of my batch jobs on Network Queueing Systems. He emailed me on Monday 19^{th} March that he had deleted one of my jobs because he thought it was doing nothing and consuming no CPU time. In his email he said it would not be able to reach its timeout limit in order to die out.

I knew what he had written could be true and little minded the fact that he killed my task before bother telling me about it first. Two minutes after having read his email I found out at the ISD department that the number was not one but all, about seven in total, of my jobs had been deleted. Zaiem Bakker agreed at that point that it was not a right thing for a system administrator to do to delete users' jobs without letting the owners know first.

What happened in this case was this. Two weeks ago one of my NQS jobs correctly produced an input file of larger than 300 MB in size before it died due to lack of resource. I suddenly realised then that another one or two of my other jobs, if allowed to go on further, would produce output files of larger than 1 GB in size each. I tried to kill them using *qdel*. I found that I could not do so, so I asked Zaiem about it. For two weeks we tried everything in vain. We could not delete either one of them. Meanwhile, I did the only thing possible for me to do in order to prevent large output files from them. Not being able to kill the job, I deleted their input files instead.

It must have been for this reason that one of them went into a sleeping state as Nicholas said. But a sleeping program could go on for a long time trying to read a nonexisting file without causing much trouble or even consuming much CPU time.

When Nicholas came on the phone that morning, however, he completely changed the story. Instead of saying that one of my jobs had been sleeping, it turned out to be that my jobs had *messed* up *the whole* NQS systems. He told me not to use NQS again after this.

I hope you understand that it is one thing to say that a job sleeps; it is another to say that it disrupts. In fact I think they are as opposit to one another as *dull* and *vicious*.

I have just proved that two of my programs which Nicholas killed actually completed very quickly when run interactively. I waited while the two ran in the foreground. One (f72.m) completed within 10 minutes while the other (f71.m) took less than one hour to run.

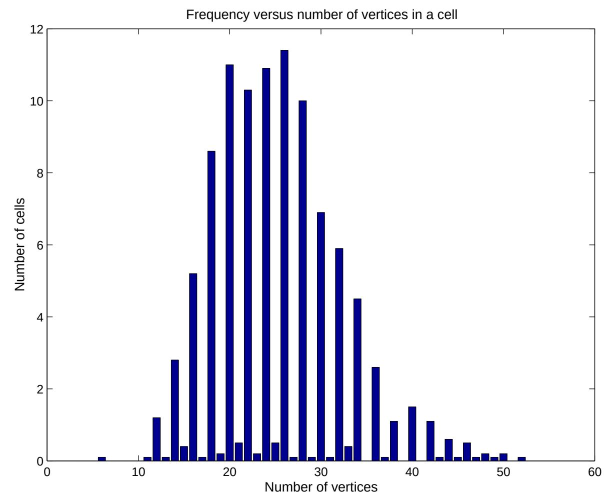
A program submitted in January and killed in February due to its idleness had not gone to sleep as Nicholas suspected and Iain believed. The problem was with NQS. It has not been set up properly and it did not stop the process when after 24 hours. Instead, it let it run until well over 13 hours CPU time without doing anything about it. That process could, however, possibly have gradually slowed down for lack of resource, but not because deligence was missing. The people who looked after NQS obviously did not even realise this fact. Lack of time, as has been said, could not be a sound excuse since this was their major job.

Some interesting facts I found out about NQS from this experience are the following. The command `qdel` can kill only jobs waiting in a queue. A running job needs to be `kill` from the platform it is running. NQS needs to be set up properly in order for it to kill a process which reaches its time limit. Matlab batch files might not expire when run on NQS. This queueing system is very convenient except that it needs absolute pathing in everything you do. Is there any way to configure NQS or write a shell script to help making it more user-friendly ?

More on number of vertices 3–d

The following was done on a 3–d systems with a joggled-input option in *qhull*. The number of vertices, however, takes into account the boundary cells as well. The minimum $n_{v,c}$ was 6, maximum $n_{v,c}$ 52, $\bar{n}_{v,c}$ 25.269, 2^{nd}–Moment 49.363, 3^{rd}–Moment 228.87, 4^{th}–Moment 8,704.9, $\sigma^2_{n_{v,c}}$ 49.412, $\sigma_{n_{v,c}}$ 7.0294, $\mu_{g,n_{v,c}}$ 24.317, $\mu_{h,n_{v,c}}$ 23.365, $\widetilde{n}_{v,c}$ 24, $\delta_\mu(n_{v,c})$ 5.5428, $\kappa(n_{v,c})$ 3.5725 .

Following is a distribution graph.

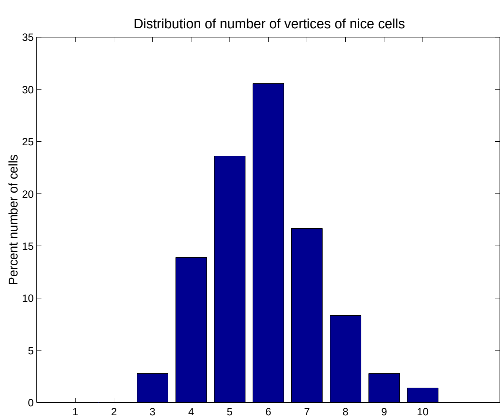


§ 2.6 Faces in different dimensions

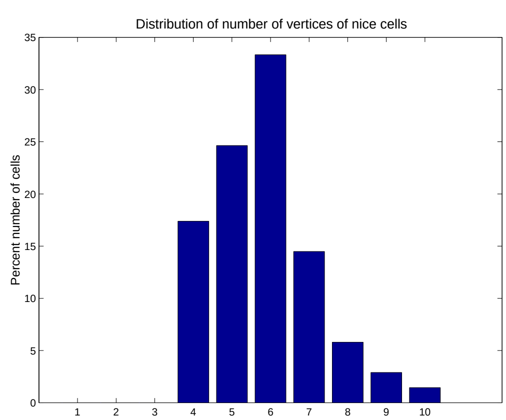
Considering only those cells bound withing the unit box, vertices and all, the following, namely Tables 2.1, 2.2 and 2.3, are the results from five simulations in two, three and four dimensions respectively, using the program listed in § 3.10.

	Random seed				
	829247	134315	67453	432243	231215
N_v	187	187	186	189	183
n_v	170	167	168	163	170
n_c	72	69	70	65	72
n_c^v	169	165	166	159	170
μ_c^v	5.875	5.8116	5.8429	5.9077	5.8194
$(\sigma_v^2)_c$	2.0264	1.8022	1.6706	1.7726	1.5022
$m^2(n_c^v)$	1.9983	1.7761	1.6467	1.7453	1.4813
$m^3(n_c^v)$	1.1263	1.6917	0.96591	0.57642	0.96103
$n_{1\mathrm{f}}$	241	235	237	227	241
$n_c^{1\mathrm{f}}$	240	233	235	223	241
$\mathrm{t_{CPU}}$(second)	20.36	20.39	20.09	20.6	19.76

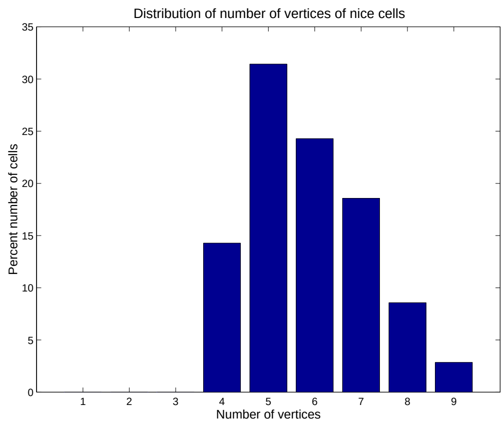
Table 2.1 *Faces of Voronoi in two dimensions.* $\mathrm{N}_c = 100$.



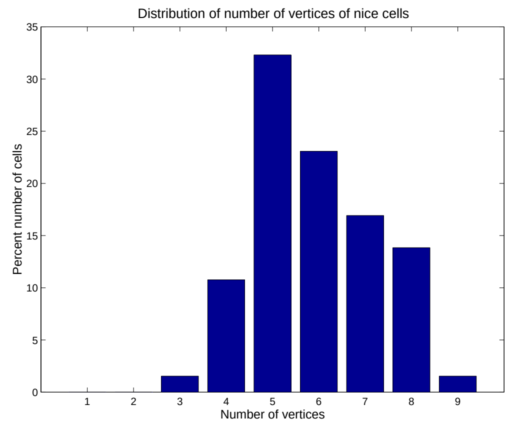
(a)



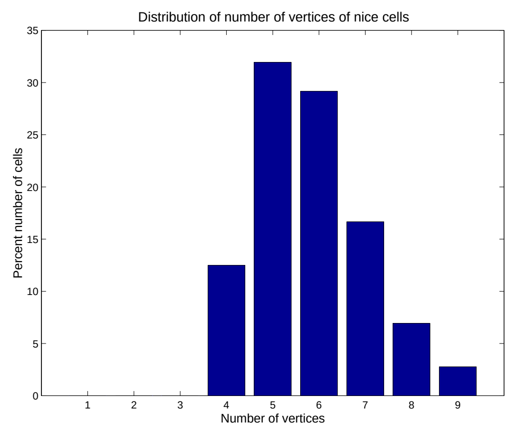
(b)



(c)



(d)



(e)

Figure 2.5 Distribution of v_c in each simulation on 2-d Voronoi.

	Random seed				
	42398198	83250	34959	743690	1321
N_v	204	221	224	225	214
n_v	148	147	143	155	146
n_c	7	5	7	8	7
n_c^v	99	79	85	109	97
μ_c^v	21.714	23.6	20.286	21	21.714
$(\sigma_v^2)_c$	4.5714	14.8	31.238	34.286	21.905
$m^2(n_c^v)$	3.9184	11.84	26.776	30	18.776
$m^3(n_c^v)$	-4.6181	-16.128	-42.402	171	69.831
$n_{2\mathrm{f}}$	114	110	111	123	113
$\mu_{2\mathrm{f}}^v$	4.9211	4.7727	4.7928	4.8293	4.7788
$(\sigma_v^2)_{2\mathrm{f}}$	1.2592	1.6268	1.5476	2.0772	1.656
$m^2(n_{2\mathrm{f}}^v)$	1.2482	1.612	1.5336	2.0603	1.6413
$m^3(n_{2\mathrm{f}}^v)$	0.16453	0.99264	1.1334	1.7676	1.1444
$n_c^{2\mathrm{f}}$	76	60	69	88	75
$(\mu_{2\mathrm{f}}^v)_c$	5.0526	5.05	4.9855	4.9545	5.0533
$(\sigma_v^2)_c^{2\mathrm{f}}$	1.1439	1.811	1.3674	2.0209	1.7268
$m^2((n_{2\mathrm{f}}^v)_c)$	1.1288	1.7808	1.3476	1.9979	1.7038
$m^3((n_{2\mathrm{f}}^v)_c)$	0.19004	0.28275	1.5224	1.4544	0.98057
$n_c^{1\mathrm{f}}$	167	133	146	187	163
$\mathrm{t_{CPU}}$(second)	24.28	25.67	28.49	34.23	26.46

Table 2.2 *Various faces of Voronoi in three dimensions.* $\mathrm{N}_c = 50$.

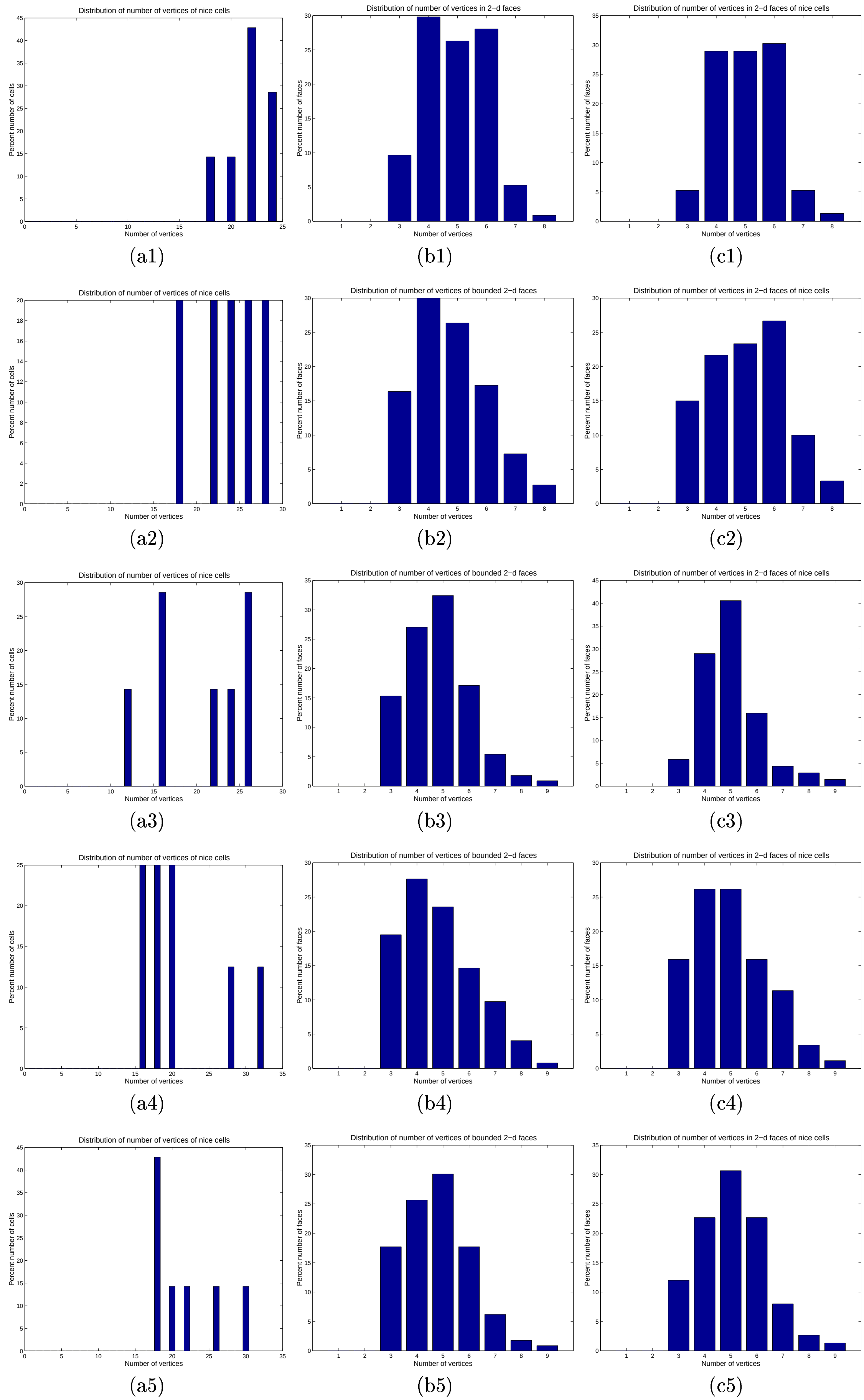
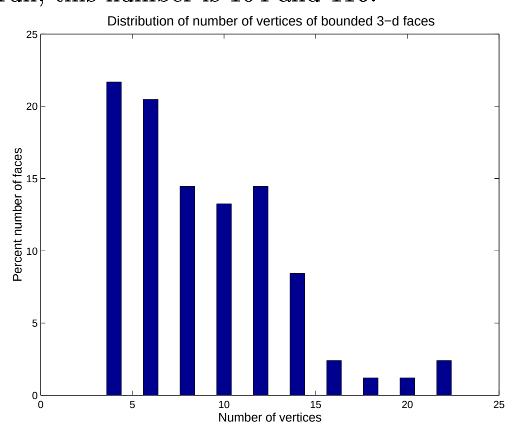


Figure 2.2 *Distribution of* (ai) v_c, (bi) $v_{2\mathrm{f}}$, *and* (ci) $v_c^{2\mathrm{f}}$ *of the* i^{th} *simulation on 3-d Voronoi.*

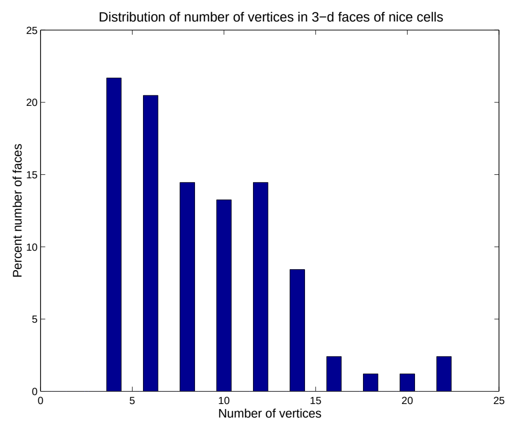
	Random seed				
	24098	802723	1453	732849	20480
N_v	1684	1719	1674	1586	1600
n_v	938	921	860	889	904
n_c	5	1	1	1	2
n_c^v	442	172	149	117	200
μ_c^v	114	172	149	117	110
$(\sigma_v^2)_c$	174.5	0	0	0	72
$m^2(n_c^v)$	139.6	0	0	0	36
$m^3(n_c^v)$	-547.2	0	0	0	0
$n_{3\mathrm{f}}$	83	102	84	83	95
$\mu_{3\mathrm{f}}^v$	8.8675	8.3137	8.5952	8.9398	8.7789
$(\sigma_v^2)_{3\mathrm{f}}$	18.848	18.198	19.449	17.496	19.77
$m^2(n_{3\mathrm{f}}^v)$	18.621	18.019	19.217	17.286	19.562
$m^3(n_{3\mathrm{f}}^v)$	75.225	102.38	95.854	62.919	118.61
$n_c^{3\mathrm{f}}$	32	8	10	7	19
$(\mu_{3\mathrm{f}}^v)_c$	8.8675	9.25	10.8	11.143	9.8947
$(\sigma_v^2)_c^{3\mathrm{f}}$	18.848	13.643	21.511	14.476	28.655
$m^2((n_{3\mathrm{f}}^v)_c)$	18.621	11.938	19.36	12.408	27.147
$m^3((n_{3\mathrm{f}}^v)_c)$	75.225	18.281	66.624	8.5364	195.1
$n_c^{2\mathrm{f}}$	11	0	5	2	7
$n_c^{1\mathrm{f}}$	2	0	0	0	1
$\mathrm{t_{CPU}}$(second)	315.07	358.94	324.56	281.85	283.82

Table 2.3 *Faces of Voronoi in four dimensions.* $\mathrm{N}_c = 100$.

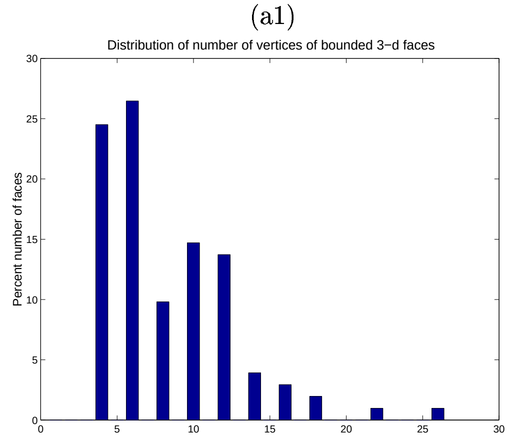
The number of vertices in each 2-d face is a constant equals to three while that of a 1-d face is two. In the first run the number of vertices in each of the cells is 95, 108, 117, 120 and 130; in the last run, this number is 104 and 116.



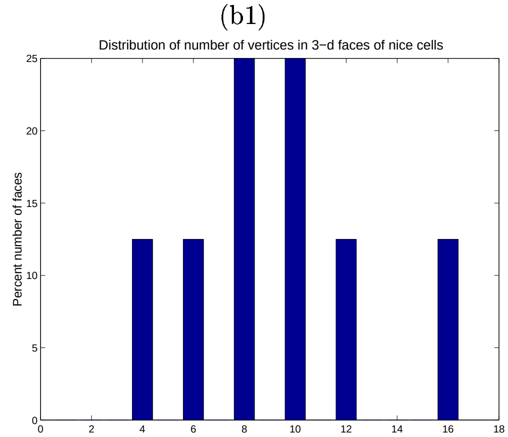
(a1)



(b1)



(a2)



(b2)

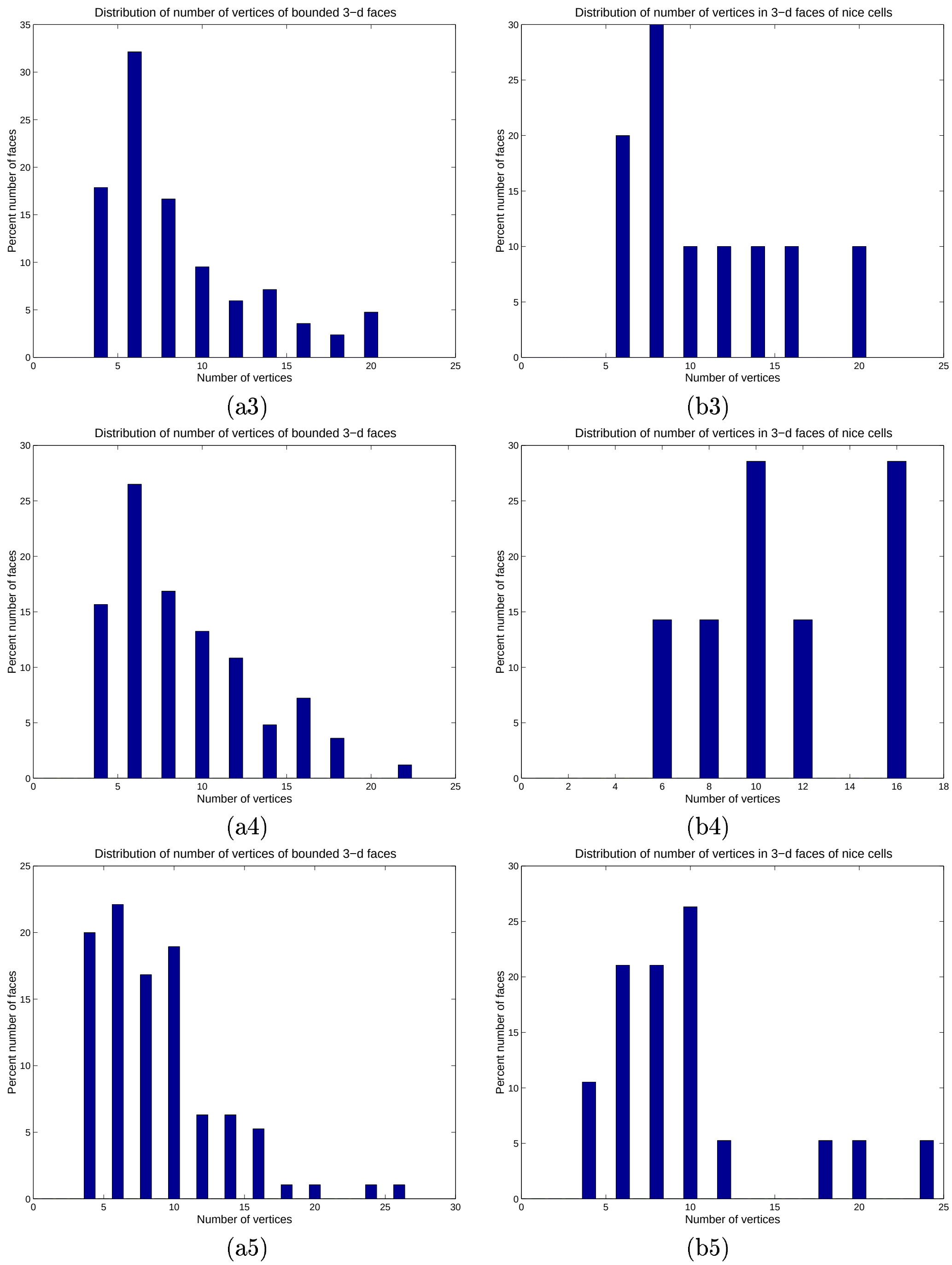
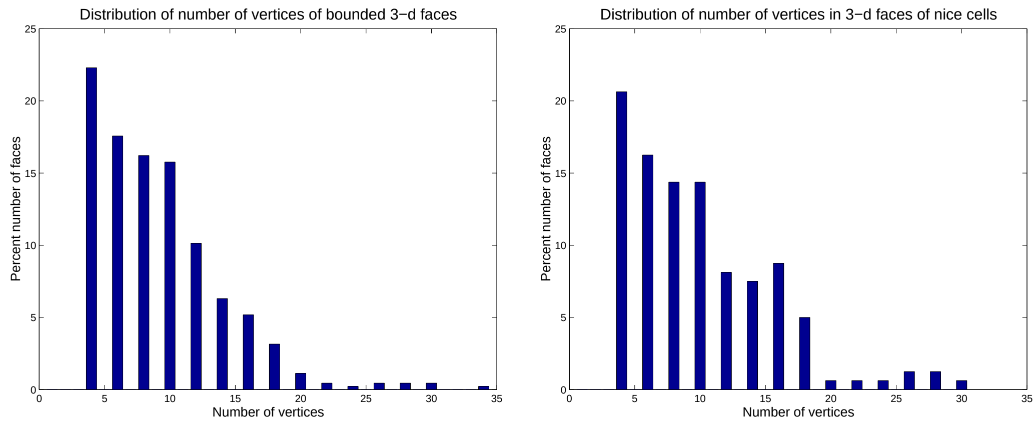


Figure 2.3 *Distribution of* (ai) $v_{3\mathrm{f}}$ *and* (bi) $v_c^{3\mathrm{f}}$ *of the* i^{th} *simulation on 4-d Voronoi.*

Table 2.4 contains the results obtained from a 4-d Voronoi network of 300 original nuclei. The numbers of vertices of the sixteen cells are 97, 99, 112, 141, 145, 160, 170, 171, 176, 176, 184, 186, 188, 192, 216 and 235.

Random seed	91876				
N_v	6776	$m^3(n_c^v)$	-1.7507×10^4	$(\mu_{3\mathrm{f}}^v)_c$	9.9875
n_v	3848	$n_{3\mathrm{f}}$	444	$(\sigma_v^2)_c^{3\mathrm{f}}$	31.1697
n_c	16	$\mu_{3\mathrm{f}}^v$	9.1486	$m^2((n_{3\mathrm{f}}^v)_c)$	30.9748
n_c^v	1687	$(\sigma_v^2)_{3\mathrm{f}}$	24.0411	$m^3((n_{3\mathrm{f}}^v)_c)$	203.3116
μ_c^v	165.5000	$m^2(n_{3\mathrm{f}}^v)$	23.9869	$n_c^{2\mathrm{f}}$	60
$(\sigma_v^2)_c$	1.5100×10^3	$m^3(n_{3\mathrm{f}}^v)$	170.4755	$n_c^{1\mathrm{f}}$	3
$m^2(n_c^v)$	1.4156×10^3	$n_c^{3\mathrm{f}}$	160	$\mathrm{t_{CPU}}$(second)	1.6013×10^4

Table 2.4 *Face statistics of 300 nuclei Voronoi in four dimensions.*



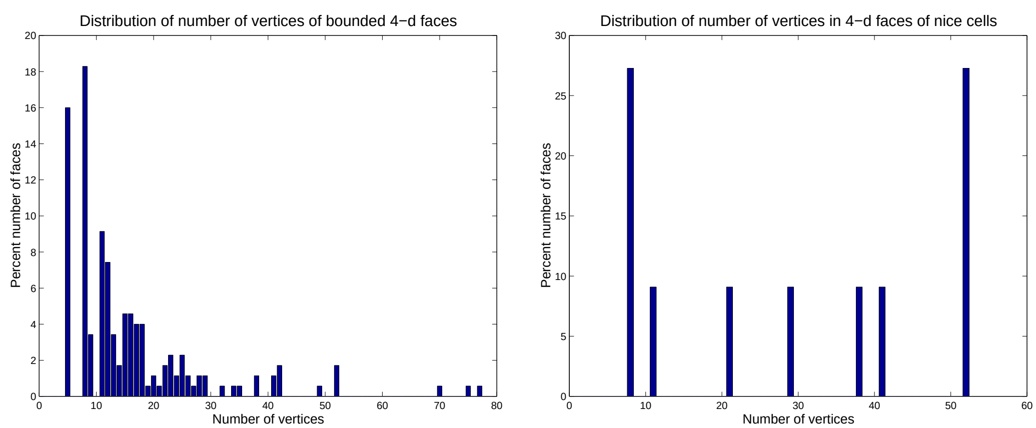
(a) (b)

Figure 2.4 *Distribution of* (a) $v_{3\mathrm{f}}$ *and* (bi) $v_c^{3\mathrm{f}}$ *of 4-d Voronoi, 300 nuclei.*

Table 2.5 is obtained from Voronoi in five dimensions.

Random seed	39378			N_v	16212
n_v	6449	$n_{4\mathrm{f}}$	175	$(\sigma_v^2)_c^{4\mathrm{f}}$	352.2909
n_c	1	$\mu_{4\mathrm{f}}^v$	15.9200	$m^2((n_{4\mathrm{f}}^v)_c)$	320.2645
n_c^v	864	$(\sigma_v^2)_{4\mathrm{f}}$	163.7752	$m^3((n_{4\mathrm{f}}^v)_c)$	351.8362
μ_c^v	864	$m^2(n_{4\mathrm{f}}^v)$	162.8393	$n_c^{3\mathrm{f}}$	1
$(\sigma_v^2)_c$	0	$m^3(n_{4\mathrm{f}}^v)$	4.8764×10^3	$(\mu_{3\mathrm{f}}^v)_c$	4
$m^2(n_c^v)$	0	$n_c^{4\mathrm{f}}$	11	$n_c^{2\mathrm{f}}$	1
$m^3(n_c^v)$	0	$(\mu_{4\mathrm{f}}^v)_c$	29.0909	$\mathrm{t_{CPU}}$(second)	1.8908×10^4

Table 2.5 *Face statistics of 200 nuclei Voronoi in five dimensions.*

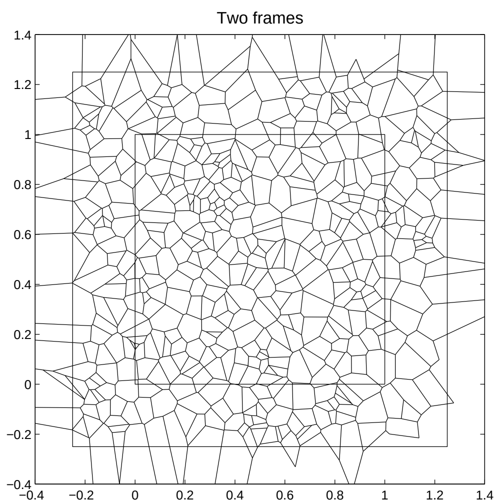


(a) (b)

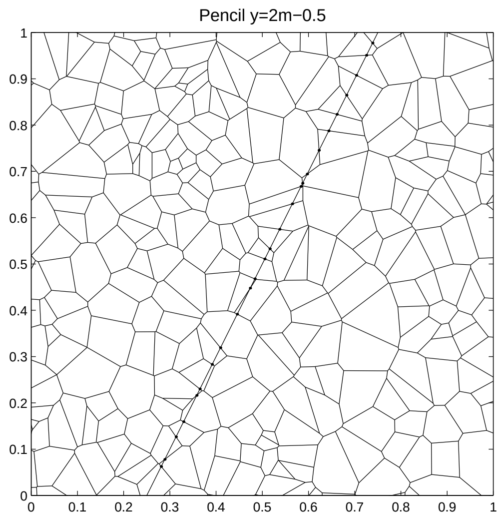
Figure 2.5 *Distribution of* (a) $v_{4\mathrm{f}}$ *and* (bi) $v_c^{4\mathrm{f}}$ *of 5-d Voronoi, 200 nuclei.*

§ 2.7 Beam intersection study

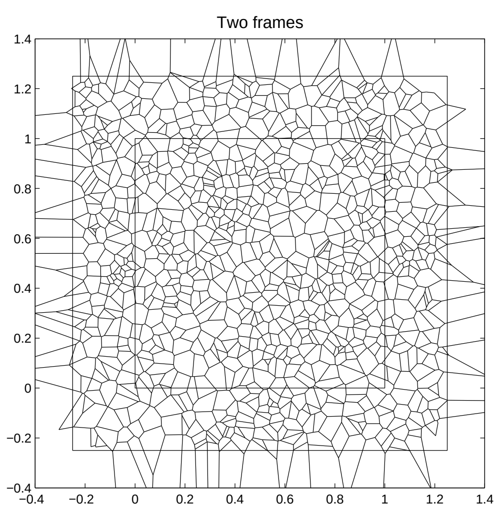
In this study of sectioning by a line the Voronoi in two dimensions, first generates on Matlab 500 points within a square box from -0.25 to 1.25 in both axes. The beam is simply a straight line $y = mx + c$ where m is the slope and c a constant.



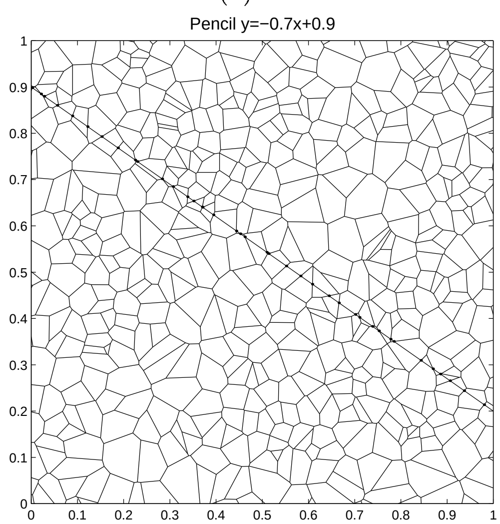
(a)



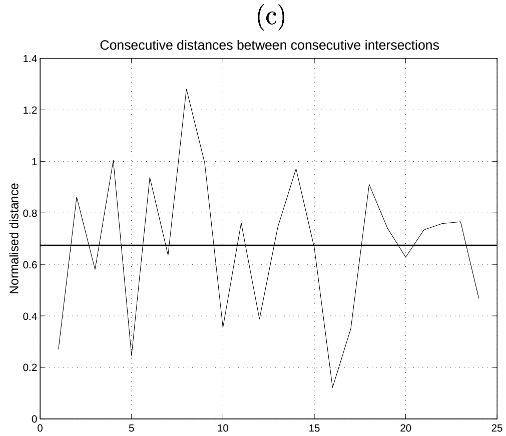
(b)



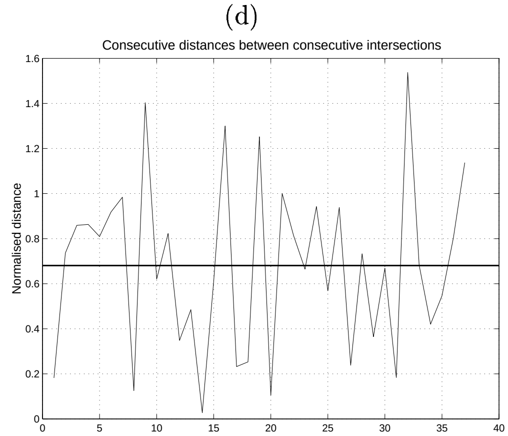
(c)



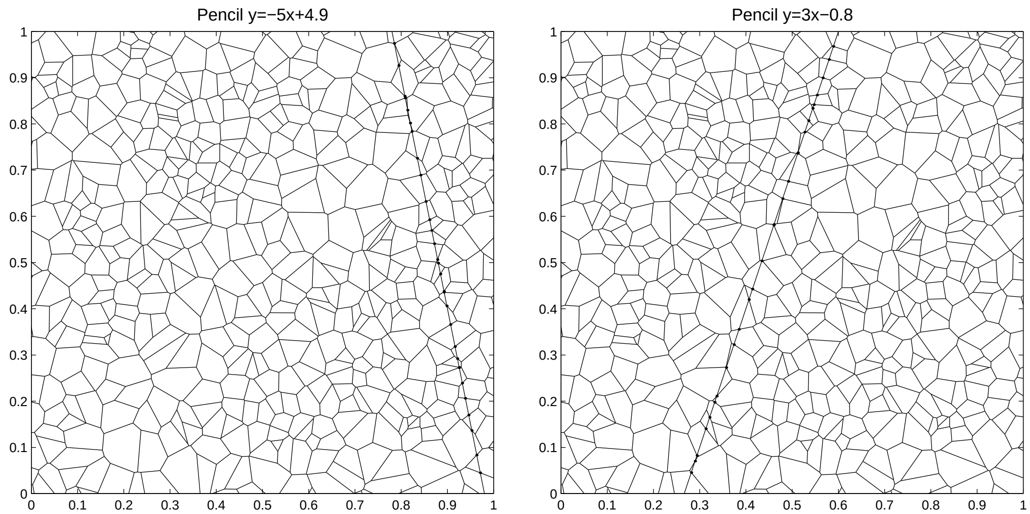
(d)



(e)



(f)



(g) (h)

Figure 2.1 *Intersection by a line.* (a) *is intersected by* (b) $y = 2x - 0.5$*;* (c) *is intersected by* (d) $y = -0.7x + 0.9$*;* (e) *and* (f) *are corresponding distances of respectively* (b) *and* (d)*;* (c) *is intersected by* (g) $y = -5x + 4.9$ *and* (h) $y = 3x - 0.8$.

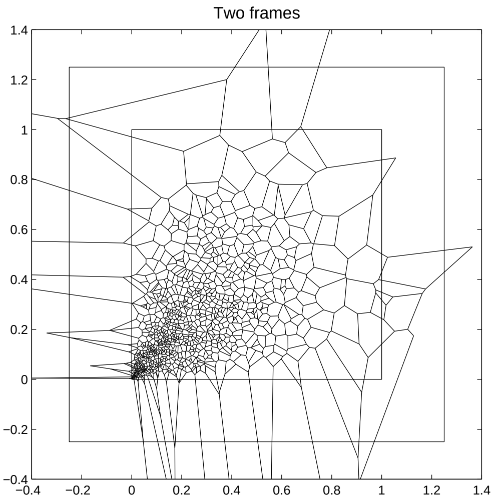
A natural basis for the normalisation is the expected distance. Another possible basis is $\frac{1}{\sqrt{239}} = 0.064685$. I call *mean normalisation* the normalisation using the first basis and *homogeneity normalisation* the second. Graphs of the closest pair distances look the same for both types of normalisation.

Simulation	1	2	3	4
N_c	500	1,000		
N_e	1,463	2,955		
n_c	239	463		
n_e	789	1,455		
Line equation	$y = 2x - 0.5$	$y = -0.7x + 0.9$	$y = -5x + 4.9$	$y = 3x - 0.8$
$\bar{d}$	4.6867×10^{-2}	3.163×10^{-2}	3.2448×10^{-2}	3.7843×10^{-2}
σ_d^2	3.9422×10^{-4}	3.1174×10^{-4}	2.9005×10^{-4}	4.9535×10^{-4}
Normalisation				
by mean	1 ± 0.4237 0.1713 -8.1332×10^{-3}	1 ± 0.5582 0.3032 3.5783×10^{-2}	1 ± 0.5249 0.2656 -3.7762×10^{-3}	1 ± 0.5881 0.3321 2.2989×10^{-2}
by homogeneity	0.7245 ± 0.3070 8.9936×10^{-2} -3.0936×10^{-3}	0.68059 ± 0.3799 0.1404 1.128×10^{-2}	0.6982 ± 0.3665 0.1295 -1.2853×10^{-3}	0.81428 ± 0.4789 0.2202 1.2412×10^{-2}

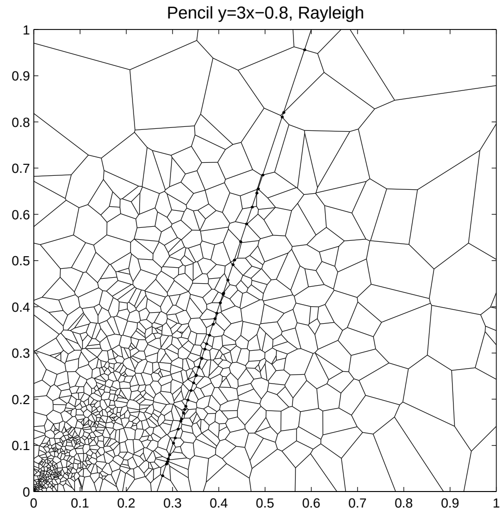
The program used is listed in § ℵ. The space position vector of the intersection between the two vectors AB and CD is $P = A + r(B - A)$, where $AB = B - A$ and $CD = D - C$ are vectors or directed lines and A, B, C, D are space vectors, $r = \begin{vmatrix} CD' \\ CA' \end{vmatrix} \Big/ \begin{vmatrix} AB' \\ CD' \end{vmatrix}$ and $s = \begin{vmatrix} AB' \\ CA' \end{vmatrix} \Big/ \begin{vmatrix} AB' \\ CD' \end{vmatrix}$ Here $AB = A + r(B - A)$, and $CD = C + s(D - C), 0 \le r, s \le 1$ are directed lines. P exists if $0 \le r \le 1$ and $0 \le s \le 1$.

If the denominator $\begin{vmatrix} AB' \\ CD' \end{vmatrix}$ is zero, then the two lines are parallel. Also, if the nominator of r is zero, that is $\begin{vmatrix} CD' \\ CA' \end{vmatrix} = 0$, then both lines are collinear.

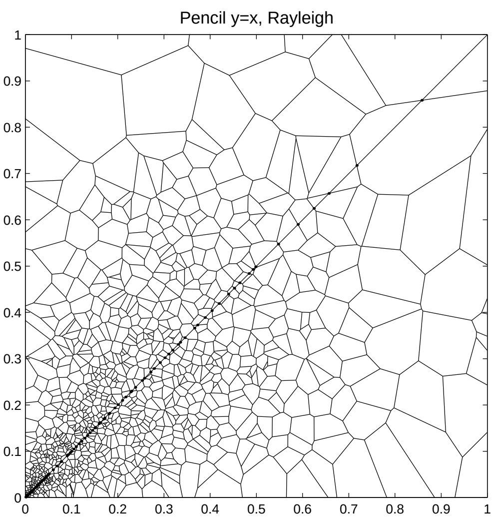
Consider the line section of Rayleigh distributed Voronoi where both the coordinates x and y are random numbers with Rayleigh distribution. The probability density function of the Rayleigh distribution is $y = f(x/b) = \frac{x}{b^2} e^{\frac{-x^2}{2b^2}}$. The mean of this distribution is $b\sqrt{\frac{\pi}{2}}$ and the variance is $\frac{4-\pi}{2} b^2$. With $b = 1, 2, \ldots, 1000$ choose the random numbers from the Rayleigh distribution, then scale and use them as the coordinates.


Two frames

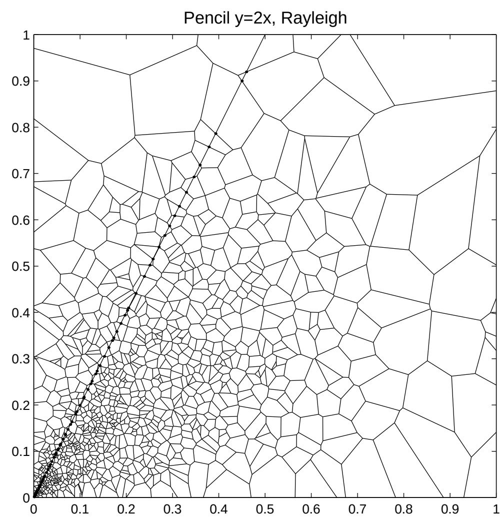
(a)


Pencil y=3x−0.8, Rayleigh

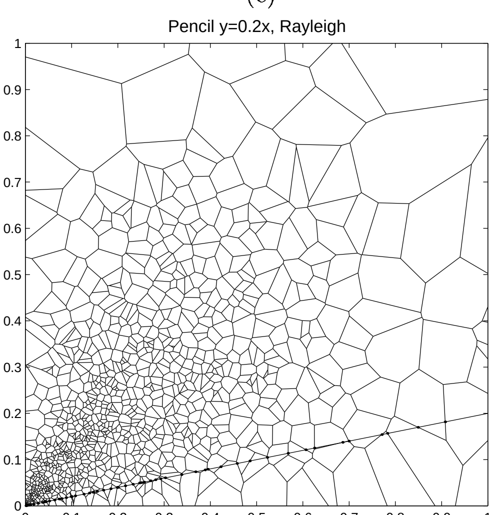
(b)


Pencil y=x, Rayleigh

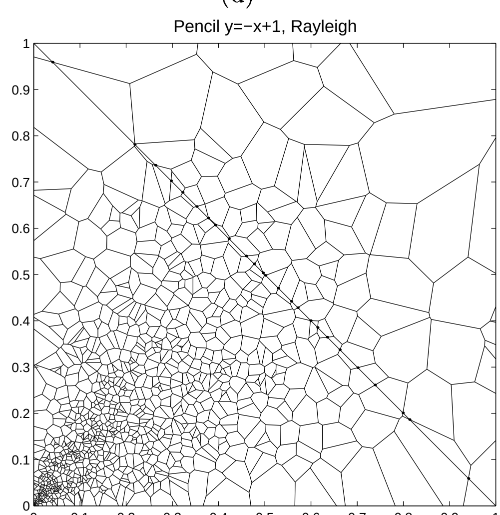
(c)


Pencil y=2x, Rayleigh

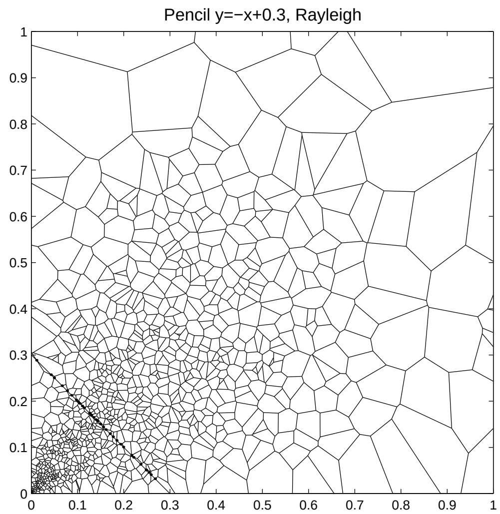
(d)


Pencil y=0.2x, Rayleigh

(e)


Pencil y=−x+1, Rayleigh

(f)



(g)

Figure 2.2 Line intersection in Rayleigh distributed Voronoi. The structure in (a) is intersected by (b) $y = 3x - 0.8$, (c) $y = x$, (d) $y = 2x$, (e) $y = 0.2x$, (f) $y = -x + 1$, (g) $y = -x + 0.3$

Simulation	1	2	3	4	5	
N_c	1,000					
N_e	2,975					
n_c	988					
n_e	2,912					
Line eq.	$y = 3x - 0.8$	$y = x$	$y = 2x$	$y = 0.2x$	$y = -x + 1$	$y = -x + 0.3$
$\bar{d}$	2.3621×10^{-2}	1.4253×10^{-2}	1.3853×10^{-2}	1.7434×10^{-2}	4.4401×10^{-2}	1.027×10^{-2}
σ_d^2	8.1582×10^{-4}	6.2922×10^{-4}	2.9647×10^{-4}	3.0991×10^{-4}	1.1584×10^{-3}	3.5365×10^{-5}
Normalised						
by mean	1 ± 1.2092 1.4256 5.5516	1 ± 1.7599 3.0609 28.351	1 ± 1.2429 1.5239 7.6499	1 ± 1.0098 1.0004 1.6643	1 ± 0.7665 0.56204 1.2114	1 ± 0.5791 0.3245 0.1755
by homog.	0.7425 ± 0.8978 0.7859 2.2721	0.4480 ± 0.7885 0.6144 2.5493	0.4354 ± 0.5412 0.2890 0.6316	0.548 ± 0.5533 0.3004 0.2739	1.3956 ± 1.0698 1.0947 3.293	0.3228 ± 0.1869 3.3813×10^{-2} 5.9021×10^{-3}

Table 2.2 *Line intersection statistics of Rayleigh distributed Voronoi*

The following codes generate and scale the points of Rayleigh distribution.

```
X=raylrnd([1:NumCell])';Y=raylrnd([1:NumCell])';Max=0.8*max([X;Y]);X=X/Max;Y=Y/Max;
```

The values of Cx and Dx will need to be adjusted manually from the pencil beam equation. It is the value of x at both points of intersection between the beam and the $[0, 1]$ square box. A random Voronoi network can either be both homogeneous and isotropic or, nonhomogeneous and nonisotropic depending on whether the probability distribution function is a constant.

3
Percolation

§ 3. Percolation

The 1970's and the 1980's saw a proliferation of variations on the theme of percolation. Every year there seemed to be a new percolation problem or two. For a *Bethe lattice* Chalupa *et al* (1979) reported a *bootstrap percolation* where those randomly occupied sites with less than m occupied neighbours are recursively emptied one by one until a stable configuration is reached. The problem they are interested in is that where the impurity concentration, dilution and crystal-field interaction compete in magnetic materials compete against the exchange interaction, resulting in the magnetic moments and consequently the magnetic order being destroyed.

Wilkinson and Willemsen (1983) introduced *invasion percolation.* Working with Schlumberger they were interested in the real problem in the oil industry where water displaces oil via capillary action. Their approach was that of a constant flow rate, not one of a constant pressure as usual. Here water displaces oil in the smallest available pores. But when water completely surrounds any region of oil, no further advance into that region will be possible, oil being incompressible. Such regions are called *trappings* and cause a problem generally known by the name of *residual oil*, an economic bane for oil industry.

In the above example the hydrophobic versus hydrophilic propertyplays an important role in the replacement of oil in pores with water. And water is prevented from penetrating trapped oil regions by much the same principle as that which prevents the water in the contents of a sandwich from crossing the spreaded layer of butter to wet the hydrophilic bread. For this latter case the pores in question are those within the bread texture, and the soaking of the bread is best prevented provided that the trapped regions of butter or margarine percolates in two dimensions to form a single layer or film which entirely covers the sectioning surface of the bread. Moreover, there is a similarity also in the internal structure of both the rock and the bread. The density of pores in bread is not homogeneous as a result of the tension on the surface of the dough caused by internal air pressure originated from the yeasts inside, as well as because of the heat applied to it when inside the oven, which dehydrolises the surface and makes it dry and hardened. The same inhomogeneity can be found inside the rocks which form the oil reservoirs where the regions of oil are surrounded by rocks which are less porous and have less permeability.

For Adler and Aharony (1988) a random walker, aka ant, treads on clusters. The ant enlarges a cluster by stepping on to an empty site next to it which meets certain conditions. They called this problem *diffusion percolation.* An example of a condition met is where the empty site has two or more occupied neighbours on a square lattice.

Most percolation in studies happens by randomly toggling the phases of sites or bonds in regular lattices. Kerstein (1983) considered randomly located spheres, take the complementary region of their union, and then perform percolation on the former. He showed that such percolation problem is equivalent to a percolation on a Voronoi lattice whose sites are the sphere centres.

In the same way as an infinite loop in computer science means that one can not come to the termination of a program going along the time dimension, an infinite cluster in percolation means that one can not come to the end of a cluster shifting along either one of the dimensions. The former case is along the one dimension of time and is only possible because of the time flows in one direction. Therefore half the line spanned is considered as being infinite. In one nondirectional dimension, except for the trivial case where one can consider the entire space as being one single cluster, no infinite cluster is possible. In simulation, when a cluster spans the whole of the space considered along any dimension we say that it is infinite since as one moves a cross section along that dimension, it always contains a section of the cluster.

The bond percolation program of two-dimensional Voronoi network by Tiyapan (1995, p. 78) takes $\mathrm{O}(n^2)$ time provided we assume that the contribution made by the number of clusters together with the that by each cluster comparison amount to a linear time complexity term. In order to justify this assumption, consider first the number of clusters. The maximum number of clusters possible depends on the size of the system, in other words it must vary as n_1^k, where $0 < k_1 < 1$. This maximum value should be approximately 0.5 because of symmetry between occupansy and void clusters. Next consider the time involved in comparing two clusters. On Matlab this is a sparse vector comparison which is likely to involves some linked lists, and similarly should take time as a function of n_2^k, $0 < k_2 < 1$. Because on average the size of clusters is always small before Pc, k_2 will be less than 0.5. Therefore $n_1^k \cdot n_2^k < n$ and it is safe to assume $\mathrm{O}(n)$ time from both of them combined. *q.e.d.*. A C translation (Tiyapan 1995, p. 80) of this program, though not as bad as it may seem because constant coefficients are small, gives $O(n^5)$ time in comparison.

In the field of geology Miller *et al* (2000) studies the analogy between the dilatant slip in

earthquakes and the hydrofraction occurred in melting and dehydration, the percolation of the latter in the permeability network internal to the fault being the cause of the former. According to them, the existence of toggle switches in the permeability rules out the assumption that the permeability throughout the whole system is homogeneous. Instead, the system reorganises itself into sytems of different scales of interaction according to the degree and nature of its inhomogeneity. At the critical state the scale of interaction is equal to the scale of the model.

The percolation probabilities given by Stauffer and Aharony (1994), the first number being for sites and the second for bonds, are for the honey comb lattice 0.6962, 0.65271; square 0.592746, 0.50000; triangular 0.50000, 0.34729; diamond 0.43, 0.388; simple cubic 0.3116, 0.2488; body-centred cubic 0.246, 0.1803; face-centred cubic 0.198, 0.119; 4-d hypercubic 0.197, 0.1601; 5-d hypercubic 0.141, 0.1182; 6-d hypercubic 0.107, 0.0942; 7-d hypercubic 0.089, 0.0787.

De Gennes *et al* (1959) investigated disordered binary solid solution AB where active atoms A are randomly replace the nodes of the periodic matrix B. There exists a critical concentration A in B below which all clusters are finite, and above which both finite and non-finite, *i.e.* infinite, clusters exist. Such solid solution in networks can represent the spin waves in alloys with one ferromagnetic component or the impurity bands in semiconductors. They cited seminal work on percolation by Broadbent *et al* (1957), but no mention was made about the Ising model.

In a way, percolation is similar to diffusion. In diffusion the particles considered move about randomly, whereas in percolation they can only crop up randomly at predetermined locations on a network which is fixed. We could imagine, for instance, cars running along the roads within a traffic network as diffusing through them. Then the percolation could occur on a larger scale, that is the scale of a road. The cars move along, that means they diffuse; but the roads remain fixed, and so their phases could percolate. In other words, in diffusion the particles move while in percolation, whether there are moving particles or not, it is the phases that percolate. Since historically percolation began as the study of diffusion of particles in a network of tubes in which the phases are naturally defined as the tubes being blocked or unblocked, these definitions have become most frequently used in other areas of application, for example in filtering membranes and traffic networks.

But this is not necessarily the case. Instead of dealing with a fixed network, one may consider a model of percolation in a continuum, for example by randomly patching an area until all the patches connect with one another somehow and percolate. The patch could be of any shape, as well as polygonal and circular. We can consider the percolation in a certain area as having occurred when there appears a cluster of patches which traverses any two opposite sides. One application of this is in the study of occurences of epidemics. Hoyle and Wickramasinghe (1979), having given a convincing argument in favour of viruses and various forms of diseases being carried to Earth from space by comets, talk about patchiness of pathogenic clouds. According to them, simultaneous attacks across vast region rules out person-to-person theory. Moreover, influenza epidemics are generally characterised by sudden onsets and equally sudden ends. These epidemics and plagues may be thought of as the percolation of these patches in a sufficiently large, predefined area. The bacteria and viruses coming from space adding to gene give the possibility of jump patterns in evolution (*cf* Smith and Szathmáry, 1995). The cells deliberately refuse to block viruses because they could prove to be useful in the long run, generally not by individuals but by the species. Historical examples are the disease described by Thucydides between 431 and 404 BC, five epidemics of 'English Sweats' between 1485 and 1552, and more than ten influenza pandemic from 1700 to 1900. Example of diseases which are caused by bacteria and viruses are bubonic plague, chicken pox (varicella), cholera, common cold, Legionnaires' disease, leprosy, measles, mumps, poliomyelitis, small pox, tuberculosis, and trachoma. Examples of major evolutionary transitions are those going from RNA to DNA, from prokaryotes to eukaryotes, and from axexual cloning to sexual propagation. Another example is the transition from primate to human both of whom differ from each other neurologically in the ability to use language and the power to conceptualise. One description of the Great Plague in London (Dickens, 1851). There was a rumour that a few people died in the winter of 1664. In May 1665 the disease burst out in St. Giles's which raged through July and September in every part of England; approximately 10,000 people died in London alone. But then the equinox winds virtually blew the disease away, and the Plague quickly disappeared. The existence of interstellar organic matters is supported by strong evidences with more and more complex substances constantly found (*cf* Hoyle and Wickramasinghe, 1978).

The growth mechanism of clusters in percolation is all so found in biology. Williams and Bjerknes (1972) simulate a tumour in the basal layer of an epithelium. The basal cells become less sensitive to Charlone which controls cellular division, and thus they divide κ times faster than the

normal cells where $\kappa > 1$ is the carcinogenic advantage. Abnormal cells interior to the basal layer divide, push, and then replace the neighbouring cells leaving the overall configuration unchanged except at the border where the abnormal cells exert a thrust of $\kappa - 1$ on their normal neighbours, that is $\frac{\mathrm{d}N}{\mathrm{d}t} = (\kappa - 1)n$ where n and N are respectively the numbers of peripheral and total abnormal cells. They found that dimensionality of fractal is involved and the dimension 1.1, instead of 1, must be assigned to the periphery, which means that n is proportional to $N^{0.55}$ not $N^{0.5}$. They found that abnormal cells push out faster in this order: the triangular, square, and hexagonal lattice. We may explain this, by looking at the coordination numbers of these three lattices, that the higher coordinaition number the lattice sites have, the slower the cluster expands. Added coordination means a higher degree-of-freedom the newly divided cells have to move about while still remaining local.

There is no percolation in one dimension because in such case the percolating cluster must necessarily contain the whole space. But there are some applications where the critical blockage is important, for example the blockage of drainage grilles by pea shingle is one-dimensional. Here the critical amount of blockage depends on the critical rate of flow which in turn depends on the amount of water and the rate of accumulation of water to be drained. The maintenance of gullies and grilles is done by cleaning, flushing and grit bucket emptying (*cf* Harrison and Trotman, 2002).

§ 3.1 Voronoi percolation, 2-d

The percolation of Voronoi tessellation in two dimensions can be achieved by the following algorithm.

[generate] random points; [generate] Voronoi tessellation and Delaunay triangulation; [find] vertices within the square box bounded by ı[lower] and ı[upper bounds]; [find] neighbours of all cells, bonds, vertices, and edges; **for ı[unit]** = **ı[cell, bond, vertice]** and **ı[edge] do** **find ı[number of units]**; **permute ı[list of units]**; **clear ı[cluster list 1, cluster list 2, set of resultant clusters]**; **ı[cluster percolated]** ← false; **for** $i = 1$ **to ı[number of units] do** **ı[existing cluster joined]** = false; **for** $j = 1$ **to ı[number of clusters in cluster list 1] do** **if ı[the]** j^{th} **ı[cluster]** contains **ı[the]** i^{th} **ı[unit in permuted list] do** **merge ı[the]** i^{th} **ı[unit] into ı[the]** j^{th} **ı[cluster] ;** **ı[existing cluster joined]** ← true; **end** **if ı[existing cluster joined]** is true **move ı[the]** j^{th} **ı[cluster of cluster list 1] to ı[cluster list 2] ;** **for** $k = 1$ **to ı[number of clusters in cluster list 1] do** **if ı[the]** k^{th} **ı[cluster in cluster list 1]** touches **ı[the cluster in cluster list 2] do** **merge** the former **into** the latter; **else** **append** the former **to ı[cluster list 2];** **end** **end** **test** percolation of the cluster just updated; **move ı[cluster list 2] to ı[cluster list 1];** **clear ı[cluster list 2];** **break;** **end** **end** **if ı[existing cluster joined]** is false **create** a new cluster of size one and **append** it **to ı[cluster list 1];** **end** **append ı[cluster list 1] to ı[set of resultant clusters] ;** **end end**

Algorithm 3.1 *Network percolation in 2-d.*

Pc probabilities are found to be ${}^{Vn\ 2\text{-}d}_{4(1),14(2)}p_c = 0.5110 \pm 0.0856$, ${}^{Vn\ 2\text{-}d}_{17(2)}p_b = 0.3095 \pm 0.0523$, ${}^{Vn\ 2\text{-}d}_{16(2)}p_v = 0.7231 \pm 0.0616$ and ${}^{Vn\ 2\text{-}d}_{16(2)}p_e = 0.6801 \pm 0.0468$.

And coordination numbers are ${}^{Vn\ 2\text{-}d}_{2(1),7(2)}x_c = 5.2320 \pm 0.2436$, ${}^{Vn\ 2\text{-}d}_{8(2)}x_b = 9.5022 \pm 0.2979$, ${}^{Vn\ 2\text{-}d}_{8(2)}x_v = 2.8617 \pm 0.0259$ and ${}^{Vn\ 2\text{-}d}_{8(2)}x_e = 3.8064 \pm 0.0382$.

§ 3.2 Voronoi percolation, 3-d

The following is the algorithm used for Voronoi Pc in in three dimensions.

The resulted Pc probabilities from simulation are ${}^{Vn\ 3\text{-}d}_{12(2)}p_c = 0.2340 \pm 0.0448$, ${}^{Vn\ 3\text{-}d}_{2(2),10(3)}p_b = 0.1178 \pm 0.0271$, ${}^{Vn\ 3\text{-}d}_{2(2),10(3)}p_v = 0.2941 \pm 0.0831$ and ${}^{Vn\ 3\text{-}d}_{12(3)}p_e = 0.4311 \pm 0.0324$.

And the coordination numbers are ${}^{Vn\ 3\text{-}d}_{6(2)}x_c = 11.1231 \pm 0.3329$, ${}^{Vn\ 3\text{-}d}_{1(2),5(3)}x_b = 23.0548 \pm 0.5996$, ${}^{Vn\ 3\text{-}d}_{1(2),5(3)}x_v = 3.6926 \pm 0.0245$ and ${}^{Vn\ 3\text{-}d}_{6(3)}x_e = 5.5002 \pm 0.0432$.

§ 3.3 Percolation, 2–d Voronoi section

§ 3.4 Network percolation

In the study of networks an important parameter is the coordination number, which is the number of neighbours of an element which in the graph theory is usually the vertex. Each vertex or site of a graph is connected to each of its neighbouring vertices by a bond, so the coordination number of a graph is the number of bonds connected to a vertex.

Clusters and their various characteristics play an important role in the study of percolation of networks. With a material science application in mind, Levy *et al* (1982) numerically represent the shape of a cluster by the *shape parameter* S, defined as $S = b/N$ or more generally $S = (1/2)\sum_{i=1}^{z} i\nu_i / \sum_{i=1}^{z} \nu_i$ where b or i is the number of bonds and N or ν_i the number of elements in a cluster having b or i bonds respectively.

A

Programs

§ A. Programs

§ 3.5 Network percolation, two dimensions

```
% perco
clear all; St=sum(100*clock); rand('state',St); CNa=200; Dim=2; X=rand(CNa,Dim);
[Va,Ca]=voronoin(X); T=delaunayn(X); TN=size(T,1); VNa=size(Va,1); LB=0.05;
UB=0.95; IXa=zeros(VNa,1);  V=[]; Count=0; VCNa=[];
for i=1:CNa,
  VCNa=[VCNa;size(Ca{i},2)];
end
for i=1:VNa,
  if((Va(i,1)>LB & Va(i,1)<UB) & (Va(i,2)>LB & Va(i,2)<UB))
    V=[V;Va(i,:)]; Count=Count+1; IXa(i,1)=Count;
  end
end
VN=size(V,1); VCN=[]; Count=0; Xa=X; X=[];
Tmp=sparse(1,CNa);
for i=1:CNa,
  Include=1;
  for j=1:VCNa(i,1),
    if(IXa(Ca{i}(1,j),1)==0)
      Include=0;
    end
  end
  if(Include==1)
    Count=Count+1; C{Count,1}=[]; VCN=[VCN;VCNa(i,1)];
    for j=1:VCNa(i,1),
      C{Count,1}(1,j)=IXa(Ca{i}(1,j),1);
    end
    X=[X;Xa(i,:)]; Tmp(1,i)=Count;
  end
end
CN=size(C,1); T2=[]; T3=[];
for i=1:TN,
  TmpA=[];
  for j=1:3,
    if(Tmp(T(i,j)))
      TmpA=[TmpA,Tmp(T(i,j))];
    end
  end
  TmpB=size(TmpA,2);
  if(TmpB==2)
    T2=[T2;TmpA];
  elseif(TmpB==3)
    T3=[T3;TmpA];
  end
end
% for cells
B=[]; BXX=sparse(CN,CN); NeCMat=sparse(CN,CN); Count=0;
for i=1:size(T2,1),
  Count=Count+1; B=[B;[T2(i,1),T2(i,2)]]; BXX(T2(i,1),T2(i,2))=Count;
  BXX(T2(i,2),T2(i,1))=Count; NeCMat(T2(i,1),T2(i,2))=1; NeCMat(T2(i,2),T2(i,1))=1;
end
for i=1:size(T3,1),
  for j=1:Dim,
    for k=(j+1):(Dim+1),
      if(BXX(T3(i,j),T3(i,k))==0)
        Count=Count+1; B=[B;[T3(i,j),T3(i,k)]]; BXX(T3(i,j),T3(i,k))=Count;
        BXX(T3(i,k),T3(i,j))=Count; NeCMat(T3(i,j),T3(i,k))=1;
        NeCMat(T3(i,k),T3(i,j))=1;
      end
    end
  end
end
BN=Count; A=X; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N); LBc=0.2; UBc=1-LBc;
for i=1:N,
  if(A(i,1)<=LBc)
    LMat(1,i)=1;
  elseif(A(i,1)>=UBc)
    UMat(1,i)=1;
  end
end
NeMat=NeCMat; Blocked=randperm(CN);
% for bonds
NeBMat=sparse(BN,BN);
for i=1:CN,
  [a,b,c]=find(BXX(i,:)); nc=size(c,2);
  for j=1:(nc-1),
    for k=(j+1):nc,
```

```
      NeBMat(c(1,j),c(1,k))=1; NeBMat(c(1,k),c(1,j))=1;
    end
  end
end
A=B; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if((X(A(i,1),1)<=LBc) | (X(A(i,2),1)<=LBc))
    LMat(1,i)=1;
  elseif((X(A(i,1),1)>=UBc) | (X(A(i,2),1)>=UBc))
    UMat(1,i)=1;
  end
end
NeMat=NeBMat; Blocked=randperm(BN);
%for vertices
Tmp=sparse(1,VN);
for i=1:CN,
  for j=1:VCN(i,1),
    Tmp(1,C{i}(1,j))=1;
  end
end
Vv=[];
Count=0;
for i=1:VN,
  if(Tmp(1,i))
    Count=Count+1; Vv=[Vv;V(i,:)]; Tmp(1,i)=Count;
  end
end
VvN=size(Vv,1);
for i=1:CN,
  for j=1:VCN(i,1),
    Cv{i}(1,j)=Tmp(1,C{i}(1,j));
  end
end
E=[]; EVV=sparse(VN,VN); EVVv=sparse(VvN,VvN); NeVMat=sparse(VvN,VvN);
Countv=0; Count=0;
for i=1:CN,
  Tmp=[Cv{i}(1,1:VCN(i,1)),Cv{i}(1,1)];
  for j=1:VCN(i,1),
    V1=Tmp(1,j); V2=Tmp(1,(j+1));
    if(NeVMat(V1,V2)==0)
      Countv=Countv+1; NeVMat(V1,V2)=1; NeVMat(V2,V1)=1;
    end
  end
  Tmp=[C{i}(1,1:VCN(i,1)),C{i}(1,1)];
  for j=1:VCN(i,1),
    V1=Tmp(1,j); V2=Tmp(1,(j+1));
    if(EVV(V1,V2)==0)
      Count=Count+1; E=[E;[V1,V2]]; EVV(V1,V2)=Count; EVV(V2,V1)=Count;
    end
  end
end
EN=Count; A=Vv; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
LBv=2*LB; UBv=(UB-LB);
for i=1:N,
  if(A(i,1)<=LBv)
    LMat(1,i)=1;
  elseif(A(i,1)>=UBv)
    UMat(1,i)=1;
  end
end
NeMat=NeVMat; Blocked=randperm(VvN);
%for edges
NeEMat=sparse(EN,EN); MEV=sparse(EN,VN); [a,b,c]=find(EVV); nc=size(c,1);
for i=1:nc,
  MEV(c(i),a(i))=1; MEV(c(i),b(i))=1;
end
for i=1:VN,
  a=find(MEV(:,i));
  if(~isempty(a))
    TmpN=size(a,1); Tmp=[a;a(1,1)]';
    for j=1:TmpN,
      for k=(j+1):TmpN,
        NeEMat(Tmp(1,j),Tmp(1,k))=1; NeEMat(Tmp(1,k),Tmp(1,j))=1;
      end
    end
  end
end
A=E; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if((V(A(i,1),1)<=LBv) | (V(A(i,2),1)<=LBv))
    LMat(1,i)=1;
  elseif((V(A(i,1),1)>=UBv) | (V(A(i,2),1)>=UBv))
```

```
    UMat(1,i)=1;
  end
end
NeMat=NeEMat; Blocked=randperm(EN);
% perco1
clear ClusA ClusB TSeries; NClusA=0; Perco=0;
for i=1:N,
  Joined=0;
  for j=1:NClusA,
    if(ClusA{j,3}(1,Blocked(1,i))~=0)
      ClusA{j,1}=ClusA{j,1}+1; ClusA{j,2}(1,Blocked(1,i))=1;
      ClusA{j,3}=ClusA{j,3} | NeMat(Blocked(1,i),:); Joined=1;
    end
    if(Joined==1)
      for k=1:4,
        ClusB{1,k}=ClusA{j,k};
      end
      NClusB=1;
      if(j==1)
        Tmp=ClusA; clear ClusA;
        for k=1:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{(k+1),l};
          end
        end
      elseif(j==NClusA)
        Tmp=ClusA; clear ClusA;
        for k=1:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{k,l};
          end
        end
      else
        Tmp=ClusA; clear ClusA;
        for k=1:(j-1),
          for l=1:4,
            ClusA{k,l}=Tmp{k,l};
          end
        end
        for k=j:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{(k+1),l};
          end
        end
      end
      for k=1:(NClusA-1),
        if(sum(ClusA{k,2} & ClusB{1,3}) ~= 0)
          ClusB{1,1}=ClusB{1,1}+ClusA{k,1}; ClusB{1,2}=ClusB{1,2} | ClusA{k,2};
          ClusB{1,3}=ClusB{1,3} | ClusA{k,3}; ClusB{1,4}=ClusB{1,4} | ClusA{k,4};
        else
          NClusB=NClusB+1;
          for l=1:4,
            ClusB{NClusB,l}=ClusA{k,l};
          end
        end
      end
      if((sum(full(LMat & ClusB{1,2}))~=0) & (sum(full(UMat & ClusB{1,2}))~=0))
        ClusB{1,4}=1; Perco=1;
      end
      NClusA=NClusB; ClusA=ClusB; clear ClusB; break;
    end
  end
  if(Joined==0)
    NClusA=NClusA+1; ClusA{NClusA,1}=1;
    ClusA{NClusA,2}=sparse(1,Blocked(1,i),1,1,N);
    ClusA{NClusA,3}=NeMat(Blocked(1,i),:); ClusA{NClusA,4}=0;
  end
  TSeries{i,1}=ClusA; TSeries{i,2}=Perco;
end
% Reverse
Tmp=Blocked; Blocked=[];
for i=1:N,
  Blocked=[Blocked,Tmp(1,(N-i+1))];
end
% simulations
Nc=0; TSnap=[];
for i=1:N,
  if(TSeries{i,2})
    Nc=i; break;
  end
end
Pc=Nc/N; Cord=mean(sum(NeMat,2));
```

§ 3.6 Network percolation, three dimensions

```
% perco3d.m
clear all; St=sum(100*clock); rand('state',St); CNa=300; Dim=3;
X=rand(CNa,Dim); [Va,Ca]=voronoin(X); T=delaunayn(X); TN=size(T,1);
VNa=size(Va,1); LB=0.05; UB=0.95; IXa=zeros(VNa,1); VCNa=[];
for i=1:CNa,
  VCNa=[VCNa;size(Ca{i},2)];
end
MVCa=[];
for i=1:CNa,
  Tmp=ones(1,VCNa(i,1)); MVCa=[MVCa;sparse(Tmp,Ca{i},Tmp,1,VNa)];
end
Vin=zeros(1,VNa); Count=0;
for i=1:VNa,
  if((max(Va(i,:))<1) & (min(Va(i,:))>0))
    Count=Count+1; Vin(1,i)=1; IXa(i,1)=Count;
  end
end
Tmp=~Vin; Cin=ones(1,CNa);
for i=1:CNa,
  if(sum(Tmp & MVCa(i,:)))
    Cin(1,i)=0;
  end
end
C=[]; count=0; VCN=[];
for i=1:CNa,
  if(Cin(i))
    count=count+1; TmpN=size(Ca{i},2);
    for j=1:TmpN,
      C{count,1}(1,j)=IXa(Ca{i}(1,j));
    end
    VCN(count,1)=TmpN;
  end
end
CN=size(C,1); MidBCx=sparse(CNa,CNa); MidBCy=sparse(CNa,CNa);
MidBCz=sparse(CNa,CNa); BLng=sparse(CNa,CNa);
for i=1:TN,
  Tmp=[T(i,:),T(i,1)];
  for j=1:(Dim+1),
    for k=(j+1):(Dim+1),
      if((Cin(1,Tmp(1,j)) | Cin(1,Tmp(1,k))) & ~BLng(j,k))
        MidBCx(Tmp(1,j),Tmp(1,k))=(X(k,1)+X(j,1))/2;
        MidBCx(Tmp(1,k),Tmp(1,j))=(X(k,1)+X(j,1))/2;
        MidBCy(Tmp(1,j),Tmp(1,k))=(X(k,2)+X(j,2))/2;
        MidBCy(Tmp(1,k),Tmp(1,j))=(X(k,2)+X(j,2))/2;
        MidBCz(Tmp(1,j),Tmp(1,k))=(X(k,3)+X(j,3))/2;
        MidBCz(Tmp(1,k),Tmp(1,j))=(X(k,3)+X(j,3))/2;
        dx=X(k,1)-X(j,1); dy=X(k,2)-X(j,2); dz=X(k,3)-X(j,3);
        TmpA=sqrt(dx*dx + dy*dy + dz*dz);
        BLng(Tmp(1,j),Tmp(1,k))=TmpA; BLng(Tmp(1,k),Tmp(1,j))=TmpA;
      end
    end
  end
end
Fa=[]; Count=0; FaC=[];
for i=1:CNa,
  if(Cin(1,i))
    FaC{i,1}=0; FaC{i,2}=[];
  end
end
for i=1:(CNa-1),
  for j=(i+1):CNa,
    TmpA=0; TmpB=0;
    if(Cin(1,i))
      TmpA=1;
    end
    if(Cin(1,j))
      TmpB=1;
    end
    if(TmpA | TmpB)
      Tmp=MVCa(i,:) & MVCa(j,:);
      if(sum(Tmp))
        [a,b]=find(Tmp); Count=Count+1; Fa{Count,1}=size(b,2); Fa{Count,2}=b;
        Fa{Count,3}=[MidBCx(i,j),MidBCy(i,j),MidBCz(i,j)];
        if(TmpA)
          FaC{i,1}=FaC{i,1} + 1; FaC{i,2}=[FaC{i,2},Count];
          FaC{i,3}{1,1}=i; FaC{i,3}{1,2}=j;
        end
        if(TmpB)
          FaC{j,1}=FaC{j,1} + 1; FaC{j,2}=[FaC{j,2},Count];
```

```
          FaC{j,3}{1,1}=i; FaC{j,3}{1,2}=j;
        end
      end
    end
  end
end
FaN=size(Fa,1); V=[];
for i=1:VNa,
  if(Vin(1,i))
    V=[V;[Va(i,:),i]];
  end
end
VN=size(V,1); Tmp=sparse(VNa,1);
for i=1:VN,
  Tmp(V(i,4),1)=i;
end
F=Fa;
for i=1:FaN,
  for j=1:F{i,1},
    F{i,2}(1,j)=Tmp(Fa{i,2}(1,j),1);
  end
end
FN=FaN; FC=[];
count=0;
for i=1:CNa,
  if(Cin(i))
    count=count+1; FC{count,1}=FaC{i,1};
    FC{count,2}=FaC{i,2}; FC{count,3}=FaC{i,3};
  end
end
NghV=sparse(VN,VN); Tmp=F; TmpN=FN;
for i=1:TmpN,
  TmpA=Tmp{i,2}; x=[]; y=[]; z=[]; TmpB=Tmp{i,1};
  if(TmpB==3)
    for j=1:2,
      for k=(j+1):3,
        NghV(TmpA(1,j),TmpA(1,k))=1; NghV(TmpA(1,k),TmpA(1,j))=1;
      end
    end
  else
    for j=1:TmpB,
      x=[x;V(TmpA(1,j),1)]; y=[y;V(TmpA(1,j),2)]; z=[z;V(TmpA(1,j),3)];
    end
    a=y(1)*(z(2)-z(3))+y(2)*(z(3)-z(1))+y(3)*(z(1)-z(2));
    b=z(1)*(x(2)-x(3))+z(2)*(x(3)-x(1))+z(3)*(x(1)-x(2));
    c=x(1)*(y(2)-y(3))+x(2)*(y(3)-y(1))+x(3)*(y(1)-y(2));
    K=1/sqrt(a*a + b*b + c*c); Th{1}=K*a; Th{2}=K*b; Th{3}=K*c; Max=0;
    for j=1:3,
      if(Th{j}<Max)
        Max=Th{j}; jMax=j;
      end
    end
    if(jMax==1)
      p=y; q=z;
    elseif(jMax==2)
      p=x; q=z;
    else
      p=x; q=y;
    end
    t=delaunay(p,q);
    for j=1:size(t,1),
      for k=1:3,
        t(j,k)=TmpA(1,t(j,k));
      end
    end
    Nt=size(t,1); TmpC=sparse(VN,VN);
    for j=1:Nt,
      TmpT=[t(j,:),t(j,1)];
      for k=1:3,
        TmpD=sort([TmpT(1,k),TmpT(1,(k+1))]);
        TmpC(TmpD(1,1),TmpD(1,2))=TmpC(TmpD(1,1),TmpD(1,2))+1;
      end
    end
    [k,l,m]=find(TmpC);
    for j=1:size(k,1),
      if(m(j)==1)
        NghV(k(j),l(j))=1; NghV(l(j),k(j))=1;
      end
    end
  end
end
Fed=[];
```

```
for i=1:FN,
  for j=1:2,
    Fed{i,j}=F{i,j};
  end
end
for i=1:FN,
  Count=0; TmpN=Fed{i,1};
  if(TmpN>3)
    Tmp=Fed{i,2}; TmpA=Tmp(1,1); Tmp=Tmp(1,2:TmpN); TmpN=TmpN-1; Count=Count+1;
    while(TmpN)
      a=TmpA(1,Count); TmpB=[]; Found=0;
      for j=1:TmpN,
        TmpC=Tmp(1,j);
        if(NghV(a,TmpC) & ~Found)
          TmpA=[TmpA,TmpC]; Found=1;
        else
          TmpB=[TmpB,TmpC];
        end
      end
      Tmp=TmpB; TmpN=TmpN-1; Count=Count+1;
    end
    Fed{i,2}=TmpA;
  end
end
LB2=2*LB; UB2=(UB-LB);
% cells II
Tmp=ones(1,CNa);
for i=1:CNa,
  if((max(X(i,:))>UB) | (min(X(i,:))<LB))
    Tmp(1,i)=0;
  end
end
a=find(Tmp); TmpB=sparse(1,CNa); x=[];
for i=1:size(a,2),
  TmpB(1,a(1,i))=i; x(i,:)=X(a(i),:);
end
xn=size(x,1); TmpA=zeros(size(T)); TmpN=size(T,1);
for i=1:TmpN,
  for j=1:4,
    if((Tmp(1,T(i,j))))
      TmpA(i,j)=TmpB(1,T(i,j));
    end
  end
end
nghc=sparse(xn,xn);
for i=1:TmpN,
  [a,b,c]=find(TmpA(i,:)); TmpB=size(c,2);
  if(TmpB>1)
    for j=1:(TmpB-1),
      for k=(j+1):TmpB,
        nghc(c(1,j),c(1,k))=1; nghc(c(1,k),c(1,j))=1;
      end
    end
  end
end
A=x; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if(A(i,1)<=LB2)
    LMat(1,i)=1;
  elseif(A(i,1)>=UB2)
    UMat(1,i)=1;
  end
end
NeMat=nghc; Blocked=randperm(xn);
% bonds II
[a,b,c]=find(triu(nghc)); b=[a,b]; bn=size(b,1); Tmp=sparse(bn,xn);
for i=1:bn,
  Tmp(i,b(i,1))=1; Tmp(i,b(i,2))=1;
end
nghb=sparse(bn,bn);
for i=1:xn,
  a=find(Tmp(:,i));
  if(~isempty(a))
    TmpN=size(a,1);
    for j=1:(TmpN-1),
      for k=(j+1):TmpN,
        nghb(a(j,1),a(k,1))=1; nghb(a(k,1),a(j,1))=1;
      end
    end
  end
end
A=b; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
```

```
for i=1:N,
  if((x(A(i,1),1)<=LB2) | (x(A(i,2),1)<=LB2))
    LMat(1,i)=1;
  elseif((x(A(i,1),1)>=UB2) | (x(A(i,2),1)>=UB2))
    UMat(1,i)=1;
  end
end
NeMat=nghb; Blocked=randperm(N);
% vertices II
[a,b,c]=find((NghV)); Tmp=sparse(1,VN);
for i=1:size(a,1),
  Tmp(1,b(i,1))=1;
end
d=find(Tmp); TmpN=size(d,2);
for i=1:TmpN,
  Tmp(1,d(1,i))=i;
end
for i=1:size(a,1),
  a(i,1)=Tmp(1,a(i,1)); b(i,1)=Tmp(1,b(i,1));
end
nghv=sparse(a,b,c,TmpN,TmpN); TmpA=[];
for i=1:TmpN,
  TmpA(Tmp(1,d(i)),:)=V(i,1:3);
end
A=TmpA; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if(A(i,1)<=LB2)
    LMat(1,i)=1;
  elseif(A(i,1)>=UB2)
    UMat(1,i)=1;
  end
end
NeMat=nghv; Blocked=randperm(N);
% edges II
[a,b,c]=find(triu(NghV)); E=[a,b]; EN=size(E,1); Tmp=sparse(EN,VN);
for i=1:EN,
  Tmp(i,E(i,1))=1; Tmp(i,E(i,2))=1;
end
NghE=sparse(EN,EN);
for i=1:VN,
  a=find(Tmp(:,i));
  if(~isempty(a))
    TmpN=size(a,1);
    for j=1:(TmpN-1),
      for k=(j+1):TmpN,
        NghE(a(j,1),a(k,1))=1; NghE(a(k,1),a(j,1))=1;
      end
    end
  end
end
A=E; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if((V(A(i,1),1)<=LB2) | (V(A(i,2),1)<=LB2))
    LMat(1,i)=1;
  elseif((V(A(i,1),1)>=UB2) | (V(A(i,2),1)>=UB2))
    UMat(1,i)=1;
  end
end
NeMat=NghE; Blocked=randperm(N);
% percolation
clear ClusA ClusB TSeries; NClusA=0; Perco=0;
for i=1:N,
  Joined=0;
  for j=1:NClusA,
    if(ClusA{j,3}(1,Blocked(1,i))~=0)
      ClusA{j,1}=ClusA{j,1}+1; ClusA{j,2}(1,Blocked(1,i))=1;
      ClusA{j,3}=ClusA{j,3} | NeMat(Blocked(1,i),:); Joined=1;
    end
    if(Joined==1)
      for k=1:4,
        ClusB{1,k}=ClusA{j,k};
      end
      NClusB=1;
      if(j==1)
        Tmp=ClusA; clear ClusA;
        for k=1:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{(k+1),l};
          end
        end
      elseif(j==NClusA)
        Tmp=ClusA; clear ClusA;
```

```
        for k=1:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{k,l};
          end
        end
      else
        Tmp=ClusA; clear ClusA;
        for k=1:(j-1),
          for l=1:4,
            ClusA{k,l}=Tmp{k,l};
          end
        end
        for k=j:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{(k+1),l};
          end
        end
      end
      for k=1:(NClusA-1),
        if(sum(ClusA{k,2} & ClusB{1,3}) ~= 0)
          ClusB{1,1}=ClusB{1,1}+ClusA{k,1}; ClusB{1,2}=ClusB{1,2} | ClusA{k,2};
          ClusB{1,3}=ClusB{1,3} | ClusA{k,3}; ClusB{1,4}=ClusB{1,4} | ClusA{k,4};
        else
          NClusB=NClusB+1;
          for l=1:4,
            ClusB{NClusB,l}=ClusA{k,l};
          end
        end
      end
      if((sum(full(LMat & ClusB{1,2}))~=0) & (sum(full(UMat & ClusB{1,2}))~=0))
        ClusB{1,4}=1; Perco=1;
      end
      NClusA=NClusB; ClusA=ClusB; clear ClusB; break;
    end
  end
  if(Joined==0)
    NClusA=NClusA+1; ClusA{NClusA,1}=1;
    ClusA{NClusA,2}=sparse(1,Blocked(1,i),1,1,N);
    ClusA{NClusA,3}=NeMat(Blocked(1,i),:); ClusA{NClusA,4}=0;
  end
  TSeries{i,1}=ClusA; TSeries{i,2}=Perco;
end
% Reverse
Tmp=Blocked; Blocked=[];
for i=1:N,
  Blocked=[Blocked,Tmp(1,(N-i+1))];
end
% simulations
Nc=0;
for i=1:N,
  if(TSeries{i,2})
    Nc=i; break;
  end
end
Pc=Nc/N; Cord=mean(sum(NeMat,2));
```

§ 3.7 Network percolation, 2–d section

```
% section
MVC=[];
for i=1:CN,
  Tmp=ones(1,VCN(i,1)); MVC=[MVC;sparse(Tmp,C{i},Tmp,1,VN)];
end
CE=[];
for i=1:EN,
  Tmp=MVC(:,E(i,1)) & MVC(:,E(i,2));
  if(sum(Tmp))
    TmpA=find(Tmp)'; TmpN=size(TmpA,2); CE{i,1}=TmpN; CE{i,2}=TmpA;
  end
end
ie=[]; je=[]; ke=[];
for i=1:EN,
  ie(i,1)=V(E(i,2),1)-V(E(i,1),1); je(i,1)=V(E(i,2),2)-V(E(i,1),2);
  ke(i,1)=V(E(i,2),3)-V(E(i,1),3);
end
a=1; b=.01; cc=0; d=-.5; v=[]; vC=[]; count=0;
for i=1:EN,
  Tmp=(a*ie(i)+b*je(i)+cc*ke(i));
  if(Tmp)
    v1=E(i,1); x1=V(v1,1); y1=V(v1,2); z1=V(v1,3); TmpA=(a*x1+b*y1+cc*z1+d);
    v2=E(i,2); x2=V(v2,1); y2=V(v2,2); z2=V(v2,3); t=-TmpA/Tmp;
    if((t>=0) & (t<=1))
      x=x1+(x2-x1)*t; y=y1+(y2-y1)*t; z=z1+(z2-z1)*t; count=count+1;
      v(count,:)=[x,y,z]; vC{count,1}=CE{i,1}; vC{count,2}=CE{i,2};
    end
  else
    if(~TmpA)  % both nom and denom = 0
      count=count+1; v(count,:)=[x1,y1,z1]; vC{count,1}=CE{i,1};
      vC{count,2}=CE{i,2}; count=count+1; v(count,:)=[x2,y2,z2];
      vC{count,1}=CE{i,1}; vC{count,2}=CE{i,2};
    end
  end
end
vn=count; cC=sparse(CN,1); count=0;
for i=1:vn,
  for j=1:vC{i,1},
    if(~cC(vC{i,2}(j)))
      count=count+1; cC(vC{i,2}(j),1)=count;
    end
  end
end
cn=count; vc=vC;
for i=1:vn,
  for j=1:vc{i,1},
    vc{i,2}(1,j)=cC(vC{i,2}(j));
  end
end
c=[];
for i=1:cn,
  c{i,1}=0; c{i,2}=[];
end
for i=1:vn,
  for j=1:vc{i,1},
    c{vc{i,2}(j),1}=c{vc{i,2}(j),1}+1; c{vc{i,2}(j),2}=[c{vc{i,2}(j),2},i];
  end
end
for i=1:cn,
  Tmp=[];
  for j=1:c{i,1},
    Tmp=[Tmp;v(c{i,2}(j),:),c{i,2}(j)];
  end
  TmpA=min(Tmp,[],1); TmpB=max(Tmp,[],1); [TmpC,TmpD]=min(TmpB-TmpA);
  if(TmpD==1)
    TmpA=Tmp(:,2); TmpB=Tmp(:,3);
  elseif(TmpD==2)
    TmpA=Tmp(:,1); TmpB=Tmp(:,3);
  else
    TmpA=Tmp(:,1); TmpB=Tmp(:,2);
  end
  TmpC=delaunay(TmpA,TmpB); TmpN=size(TmpC,1);
  for j=1:TmpN,
    for k=1:3,
      TmpC(j,k)=Tmp(TmpC(j,k),4);
    end
  end
  TmpA=sparse(vn,vn);
  for j=1:TmpN,
```

```
    for k=1:2,
      for m=(k+1):3,
        TmpA(TmpC(j,k),TmpC(j,m))=TmpA(TmpC(j,k),TmpC(j,m))+1;
        TmpA(TmpC(j,m),TmpC(j,k))=TmpA(TmpC(j,m),TmpC(j,k))+1;
      end
    end
  end
  [x,y,z]=find(TmpA); TmpB=[]; TmpC=[];
  for j=1:size(x,1),
    if(z(j)==1)
      TmpB=[TmpB;x(j),y(j)]; TmpC(y(j),1)=1;
    end
  end
  TmpA=[];
  for j=1:size(TmpC,1),
    TmpA{j,1}=[];
  end
  for j=1:size(TmpB,1),
    TmpA{TmpB(j,1),1}=[TmpA{TmpB(j,1),1},TmpB(j,2)];
  end
  Tmp=Tmp(1,4); TmpB=Tmp;
  TmpC=sparse(Tmp,1,1,vn,1); count=c{i,1}-1;
  while(count>0),
    if(~(TmpC(TmpA{Tmp}(1),1)))
      Tmp=TmpA{Tmp}(1); TmpB=[TmpB,Tmp]; TmpC(Tmp,1)=1;
    else
      Tmp=TmpA{Tmp}(2); TmpB=[TmpB,Tmp]; TmpC(Tmp,1)=1;
    end
    count=count-1;
  end
  c{i,3}=TmpB;
end
for i=1:cn,
  Tmp=[0,0,0];
  for j=1:c{i,1},
    Tmp=Tmp+v(c{i,2}(j),:);
  end
  Tmp=Tmp/c{i,1}; c{i,4}=Tmp;
end
Tmp=sqrt(a*a+b*b+cc*cc); u=[a/Tmp,b/Tmp,cc/Tmp]; uzp=u; ux=[1,0,0];
Tmp=cross(u,ux); TmpA=sqrt(Tmp(1)*Tmp(1)+Tmp(2)*Tmp(2)+Tmp(3)*Tmp(3));
uyp=Tmp/TmpA; uxp=cross(uyp,uzp); R=[uxp,0;uyp,0;uzp,0;0,0,0,1];
vp=(R*[v';ones(1,vn)])'; vp=vp(:,1:2); ad=min(vp,[],1);
vp=vp-[ad(1)*ones(vn,1),ad(2)*ones(vn,1)];; cs=[];
for i=1:cn,
  Tmp=R*[c{i,4}';1]; Tmp=Tmp(1:2)'-ad; c{i,5}=Tmp; cs=[cs;Tmp];
end
LB=min(vp(:,1)); UB=max(vp(:,1)); Tmp=UB-LB; LBv=LB+0.1*Tmp;
UBv=UB-LBv; Tmp=min(cs(:,1)); LB=min(cs(:,1)); UB=max(cs(:,1));
Tmp=UB-LB; LBc=LB+0.1*Tmp; UBc=UB-LBc;
% cell
cvm=sparse(cn,vn);
for i=1:cn,
  for j=1:c{i,1},
    cvm(i,c{i,2}(j))=1;
  end
end
nghc=sparse(cn,cn);
for i=1:(cn-1),
  for j=(i+1):cn,
    Tmp=find(cvm(i,:) & cvm(j,:));
    if(~isempty(Tmp))
      TmpN=size(Tmp,2);
      if(TmpN>1)
        for k=1:TmpN,
          nghc(i,j)=1; nghc(j,i)=1;
        end
      end
    end
  end
end
N=cn; LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:cn,
  if(cs(i,1)<=LBc)
    LMat(1,i)=1;
  end
  if(cs(i,1)>=UBc)
    UMat(1,i)=1;
  end
end
NeMat=nghc; Blocked=randperm(N);
% bond
```

```
[p,q,r]=find(triu(nghc)); b=[p,q]; bn=size(b,1); bcm=sparse(bn,bn);
for i=1:bn,
  bcm(i,b(i,1))=1; bcm(i,b(i,2))=1;
end
nghb=sparse(bn,bn);
for i=1:cn,
  Tmp=find(bcm(:,i));
  if(~isempty(Tmp))
    TmpN=size(Tmp,1);
    for j=1:(TmpN-1),
      for k=(j+1):TmpN,
        nghb(Tmp(j),Tmp(k))=1; nghb(Tmp(k),Tmp(j))=1;
      end
    end
  end
end
N=bn; LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:bn,
  if((cs(b(i,1),1)<=LBc) | (cs(b(i,2),1)<=LBc))
    LMat(1,i)=1;
  end
  if((cs(b(i,1),1)>=UBc) | (cs(b(i,2),1)>=UBc))
    UMat(1,i)=1;
  end
end
NeMat=nghb; Blocked=randperm(N);
% vertice
nghv=sparse(vn,vn);
for i=1:cn,
  Tmp=[c{i,3},c{i,3}(1)];
  for j=1:c{i,1},
    nghv(Tmp(j),Tmp(j+1))=1; nghv(Tmp(j+1),Tmp(j))=1;
  end
end
LMat=sparse(1,vn); UMat=sparse(1,vn);
for i=1:vn,
  if(vp(i,1)<=LBv)
    LMat(1,i)=1;
  end
  if(vp(i,1)>=UBv)
    UMat(1,i)=1;
  end
end
N=vn; NeMat=nghv; Blocked=randperm(N);
% edge
[p,q,r]=find(triu(nghv)); e=[p,q]; en=size(e,1); evm=sparse(en,en);
for i=1:en,
  evm(i,e(i,1))=1; evm(i,e(i,2))=1;
end
nghe=sparse(en,en);
for i=1:vn,
  Tmp=find(evm(:,i));
  if(~isempty(Tmp))
    TmpN=size(Tmp,1);
    for j=1:(TmpN-1),
      for k=(j+1):TmpN,
        nghe(Tmp(j),Tmp(k))=1; nghe(Tmp(k),Tmp(j))=1;
      end
    end
  end
end
N=en; LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:en,
  if((vp(e(i,1),1)<=LBc) | (vp(e(i,2),1)<=LBc))
    LMat(1,i)=1;
  end
  if((vp(e(i,1),1)>=UBc) | (vp(e(i,2),1)>=UBc))
    UMat(1,i)=1;
  end
end
NeMat=nghe; Blocked=randperm(N);
```

§ 3.8 Number of vertices

```
% numofvertices.m
clear all; dimmin=2; dimmax=9; batches=5; dvn=[]; cpu=[];
nmax=1000; rand('state',sum(100*clock));
for i=dimmin:dimmax,
  for j=1:batches,
    n=round(nmax/i); x=rand(n,i); t=cputime; [v,c]=voronoin(x);
    cpu(i,j)=(cputime-t)/n; lend=~floor(v); hend=~(ceil(v)-ones(size(v)));
    lhend=lend & hend; in=min(lhend,[],2); dvn(i,j)=sum(in)/n;
  end
end
dvn=[(1:dimmax)',dvn]; dvn=dvn(2:dimmax,:); figure(1); clf;
for i=1:batches,
    semilogy(dvn(:,1),dvn(:,(i+1)),'.','LineWidth',2); hold on;
end
edvn=[dvn(:,1),sum(dvn(:,2:(batches+1)),2)/batches]; tmp=edvn(:,2)./exp(edvn(:,1));
A=sum(tmp)/(dimmax-1); m=[dimmin,dimmax]; semilogy(m,A*exp(m));
cpu=[(1:dimmax)',cpu]; cpu=cpu(2:dimmax,:); figure(2); clf;
for i=1:batches,
    semilogy(cpu(:,1),cpu(:,(i+1)),'.','LineWidth',2); hold on;
end
ecpu=[cpu(:,1),sum(cpu(:,2:(batches+1)),2)/batches]; tmp=ecpu(:,2)./exp(ecpu(:,1));
B=sum(tmp)/(dimmax-1); m=[dimmin,dimmax]; semilogy(m,(B/35)*(exp(1)+2).^m);
```

§ 3.9 Vertices per cell and cell ratio

```
% numveachcell.m
clear all; dimmin=2; dimmax=6; batches=5; nmax=3000;
rand('state',sum(100*clock));
for i=dimmin:dimmax,
  for j=1:batches,
    n=round(nmax*2/i); x=rand(n,i); [v,c]=voronoin(x);
    fleet{i,j,1}=v; fleet{i,j,2}=c; fleet{i,j,3}=n;
  end
end
for i=dimmin:dimmax,
  for j=1:batches,
    v=fleet{i,j,1}; c=fleet{i,j,2}; n=fleet{i,j,3}; lend=~floor(v);
    hend=~(ceil(v)-ones(size(v))); lhend=lend & hend; in=min(lhend,[],2);
    numvc=[]; vcin=[];
    for p=1:n,
      numvc=[numvc,size(c{p},2)]; flag=1;
      for q=1:numvc(p),
        if(~in(c{p}(q)))
          flag=0; break;
        end
      end
      if(flag)
        vcin=[vcin,numvc(p)];
      end
    end
    tmpn=size(vcin,2); rcin(i,j)=tmpn/n; vec(i,j)=sum(vcin)/tmpn;
  end
end
dum=rcin; str={'c_{in} / c_{all}'}; dum=dum(2:dimmax,:); tmp=[];
for i=dimmin:dimmax,
  tmp=[tmp;i*ones(batches,1),dum((i-1),:)'];
end
figure(1); semilogy(tmp(:,1),tmp(:,2),'.','LineWidth',2); hold on;
[p,s,mu]=polyfit(tmp(:,1),tmp(:,2),4); x=(dimmin:.02:dimmax)';
y=polyval(p,x,[],mu); semilogy(x,y); dum=vec; dum=dum(2:dimmax,:); tmp=[];
for i=dimmin:dimmax,
  tmp=[tmp;i*ones(batches,1),dum((i-1),dimmin:dimmax)'];
end
figure(2); semilogy(tmp(:,1),tmp(:,2),'.','LineWidth',2); hold on;
edum=[dum(:,1),sum(dum(:,2:(batches+1)),2)/batches];
tmp=edum(:,2)./exp(edum(:,1)); A=sum(tmp)/(dimmax-1); m=[dimmin,dimmax];
semilogy(m,(A/70)*(exp(1)+4).^m); xlabel('Dimension','FontSize',14);
ylabel(str,'FontSize',14);
```

§ 3.10 Face statistics in n dimensions

```
% statsgenn.m    by K N J Tiyapan, 1st July, 2001
echo off; clear all; format short g; more off;
pt1 =fopen('./v50.dat','r'); sc1 =fscanf(pt1, '%d', 4);
Dimension=sc1(1,1);  NumVAll =sc1(2,1); NumC =sc1(3,1);
sc2 =fscanf(pt1, '%f', [Dimension, NumVAll]); VerticeAll =sc2';
CVMat =sparse(NumC, NumVAll); CFrame =ones(NumC, 1); VCFrame=zeros(NumVAll,1);
VFrame=ones(NumVAll,1);
for i=1:NumC,
  sc1 =fscanf(pt1, '%d', 1);
  for j=1:sc1,
    sc2 =fscanf(pt1, '%d', 1); Num =sc2+1; CVMat(i,Num) =1;
    if ( max(abs(VerticeAll(Num, :))) > 0.5 )
      CFrame(i,1) =0; VFrame(Num,1)=0;
    end
  end
end
fclose(pt1);
for i=1:NumC,
  VInC=find(CVMat(i,:)'); NumVInC=size(VInC,1);
  if(CFrame(i,1)==1)
    for j=1:NumVInC,
      VCFrame(VInC(j,1),1)=1;
    end
  end
end
CVNiceCMat=[];
for i=1:NumC,
  if(CFrame(i,1)==1)
    CVNiceCMat=[CVNiceCMat;CVMat(i,:)];
  end
end
CNumVNiceCMat=sum(CVNiceCMat,2); NumV=sum(VCFrame);
VVCFrameMat=zeros(NumV,2); Vertice=zeros(NumV,Dimension); Count=0;
for i=1:NumVAll,
  if(VCFrame(i,1)==1)
    Count=Count+1;
    VVCFrameMat(Count,1)=i; VVCFrameMat(Count,2)=Count;
    Vertice(Count,:)=VerticeAll(i,:);
  end
end
pt3=fopen('./n50.dat','w'); pt2=fopen('./c50.dat','r'); line=fgetl(pt2);
sc1 =fscanf(pt2, '%d', 1); sc2=fscanf(pt2,'%f',[Dimension,NumC]); Cell=sc2';
fclose(pt2); CNeighCCMat=sparse(NumC, NumC); t=cputime; FVAllMat=[]; FNumVAllMat=[];
for i=1:(NumC-1),
  for j=(i+1):NumC,
    VShared=and(CVMat(i,:), CVMat(j,:)); NumShared =sum(VShared, 2);
    NumFVAllMat=size(FVAllMat,1);
    if (NumShared >= Dimension)
      CNeighCCMat(i,j) =1; CNeighCCMat(j,i) =1; Exist=0;
      for k=1:NumFVAllMat,
        MatchExistingFV=sum(and(VShared,FVAllMat(k,:)),2);
        if(MatchExistingFV>=Dimension)
          Exist=1; break;
        end
      end
      if(Exist==0)
        FVAllMat=[FVAllMat;VShared]; FNumVAllMat=[FNumVAllMat;NumShared];
      end
    end
  end
end
FVMat=[]; FNumVMat=[]; FVCFMat=[]; FNumVCFMat=[];
for i=1:NumFVAllMat,
  VThisFace=find(FVAllMat(i,:)'); NumVThisFace=size(VThisFace,1);
  IncludeMe=1; IncludeMeToo=1;
  for j=1:NumVThisFace,
    if(VFrame(VThisFace(j,1),1)==0)
      IncludeMe=0;
    end
    if(VCFrame(VThisFace(j,1),1)==0)
      IncludeMeToo=0;
    end
  end
  if(IncludeMe==1)
    FVMat=[FVMat;FVAllMat(i,:)]; FNumVMat=[FNumVMat;FNumVAllMat(i,:)];
  end
  if(IncludeMeToo==1)
    FVCFMat=[FVCFMat;FVAllMat(i,:)]; FNumVCFMat=[FNumVCFMat;FNumVAllMat(i,:)];
  end
```

```
end
NumFVMat=size(FVMat,1); NumFVCFMat=size(FVCFMat,1); FDim=Dimension-1;
fprintf(pt3,'Face dimension: %d\n',FDim); fprintf(pt3,'Number of faces: %d\n',NumFVMat);
fprintf(pt3,'Number of vertices: \n [');
for i=1:NumFVMat,
  fprintf(pt3,'%d ',FNumVMat(i,1));
  if(mod(i,10)==0)
    fprintf(pt3,'...\n');
  end
end
fprintf(pt3,']\n'); fprintf(pt3,'Number of faces of nice cells: %d\n',NumFVCFMat);
fprintf(pt3,'Number of vertices: \n [');
for i=1:NumFVCFMat,
  fprintf(pt3,'%d ',FNumVCFMat(i,1));
  if(mod(i,10)==0)
    fprintf(pt3,'...\n');
  end
end
fprintf(pt3,']\n'); DVMat=FVCFMat; NumD=NumFVCFMat;
for d=3:Dimension,
  FaceCond=Dimension-d+2; DNeighDDMat=sparse(NumD,NumD); dVMat=[]; dNumVMat=[];
  for i=1:(NumD-1),
    for j=(i+1):NumD,
      VShared=and(DVMat(i,:), DVMat(j,:)); NumShared =sum(VShared, 2);
      NumdVMat=size(dVMat,1);
      if (NumShared >= FaceCond)
        DNeighDDMat(i,j) =1; DNeighDDMat(j,i) =1; Exist=0;
        for k=1:NumdVMat,
          MatchExistingdV=sum(and(VShared,dVMat(k,:)),2);
          if(MatchExistingdV>=FaceCond)
            Exist=1; break;
          end
        end
        if(Exist==0)
          dVMat=[dVMat;VShared]; dNumVMat=[dNumVMat;NumShared];
        end
      end
    end
  end
  FDim=Dimension-d+1; fprintf(pt3,'Face dimension: %d\n',FDim);
  fprintf(pt3,'Number of faces: %d\n',NumdVMat);
  if(FDim~=1)
    fprintf(pt3,'Number of vertices: \n [');
    for i=1:NumdVMat,
      fprintf(pt3,'%d ',dNumVMat(i,1));
      if(mod(i,10)==0)
        fprintf(pt3,'...\n');
      end
    end
    fprintf(pt3,']\n');
  end
  DVMat=dVMat; NumD=NumdVMat;
  if(FDim==2)
    FVMat=DVMat;
  end
end
Time=cputime-t; NumNiceC=sum(CFrame); NumVBound=sum(VFrame);
```

B Bibliography

§ B. Bibliography

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Translation

§ C. Translation

§ 3.12 G. F. Voronoi, 1909

New application of continuous parameters to the theory of quadratic form.
Second Memoir
Studies on the primitive parallelohedra by Mr. Georges Voronoi in Warsaw
Second Part
Domains of quadratic forms corresponding to the various types of primitive parallelohedra
Section IV
Various types of primitive parallelohedra

[*Journal für die reine und angewandte Mathematik*, V. 136, p. 67–181, 1909]
[translated by K N Tiyapan]

On the number of faces in $n-1$ dimensions of primitive parallelohedron.

55 *Theorem. The number of faces in $n-1$ dimensions of a primitive parallelohedron is equal to* $2(2^n-1)$.

Let us suppose that a primitive parallelohedron R corresponding to a positive quadratic form $\sum\sum a_{ij}x_ix_j$ is defined by the independent inequalities

$$\sum\sum a_{ij}l_{ik}l_{jk} \pm 2\sum \alpha_i l_{ik} \geq 0.\ (k=1,2,\ldots,\tau) \tag{1}$$

We have seen in Number 48 that any system

$$\pm(l_{1k}, l_{2k}, \ldots, l_{nk}),\ \ (k=1,2,\ldots,\tau) \tag{2}$$

represents the minimum of the quadratic form $\sum\sum a_{ij}x_ix_j$ in the set composed of all the systems of integers which are congruent to the system $\pm(l_{ik})$ with respect to the modulus 2. The form $\sum\sum a_{ij}x_ix_j$ possesses in this set only two minimum representations $\pm(l_{ik})$.

Let us divide the set E, composed of all the systems (x_i) of integers $x_1, x_2, \ldots, x_n$, into 2^n classes

$$E_0, E_1, \ldots, E_m \ \text{ where }\ m = 2^n - 1$$

with regard to the modulus 2 and suppose that the set E_0 is composed of systems the elements of which have the common divisor 2.

All the systems (2) do not belong to the different sets

$$E_1, E_2, \ldots, E_m \ \text{ where }\ m = 2^n - 1.$$

It follows that

$$\tau \leq 2^n - 1.$$

I argue that $\tau = 2^n - 1$. Let us suppose that among the systems (2) there are not found the systems belonging to a set E and we determine the minimum of the form $\sum\sum a_{ij}x_ix_j$ in the set E_h. Let (l_i) be a representation of this minimum.

Let us indicate by

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is}) \tag{3}$$

the vertices of the parallellohedron R defined by the inequalities (1) and examine the values of the function $\sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i$ which correspond to the different vertices (3). Let us suppose that the sum $\sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i$ be the smallest one.

By virtue of the supposition made, one will have the inequalities

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_{ir}l_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i.\ (r=1,2,\ldots,s) \tag{4}$$

By noticing that each point (α_i) belonging to the parallelohedron R can be determined by the equalities

$$\alpha_i = \sum_{r=1}^{s} \vartheta_r \alpha_{ir} \text{ where } \sum \vartheta_r = 1 \,\text{and}\, \vartheta_r \geq 0, (r=1,2,\ldots,s)$$

one will deduce the inequalities (4) an inequality

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i$$

which holds for any poit (α_i) belonging to the parallelohedron R. The system (l_i) which represents the minimum of the form $\sum\sum a_{ij}x_ix_j$ in the set E_h verifies the inequality

$$\sum\sum a_{ij}x_ix_j - \sum\sum a_{ij}x_il_j \geq 0$$

in the set E. It results that the point

$$\xi_i = -\frac{1}{2}\sum_{j=1}^{n} \alpha_{ij}l_j$$

belong to the parallelohedron R. By making in the inequality (5) $\alpha_i = \xi_i$, one notices

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i \leq 0.$$

The vertex (α_{ik}) of the parallelohedron R verifies the inequality

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i \geq 0,$$

therefore it is necessary that

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i = 0. \tag{6}$$

This stated, let us notice that the vertex $(\alpha_i k)$ of the primitive parallelohedron is simple.
Let us indicate by

$$\sum\sum a_{ij}l_{ir}^{(k)}l_{jr}^{(k)} + 2\sum \alpha_{ik}l_{ir}^{(k)} = 0, \quad (r = 1, 2, \ldots, n)$$

n equations which define the vertex in the parallelohedron R. As the vertex (α_{ik}) is simple, one will have an inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_{ik}x_i > 0,$$

whatever the integer values of $x_1, x_2, \ldots, x_n$ may be, the system (0) and the systems

$$(l_{i1}^{(k)}), (l_{i2}^{(k)}), \ldots, (l_{in}^{(k)}) \tag{7}$$

being excluded. By virtue of the equality (6), the system (l_i) is found among the systems (7) which all belong to the series (R).
It is therefore demonstrated that

$$\tau = 2^n - 1$$

and that the number of faces in $n-1$ dimensions of the parallelohedron R is equal to

$$2\tau = 2(2^n - 1).$$

Definition of the type of primitive parallelohedra.

56 Let us examine a primitive parallelohedron R determined with the help of independent inequalities

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} \geq 0. \; (k = 1, 2, \ldots, \sigma \text{ where } \sigma = 2(2^n - 1))$$

Let us indicate with

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is})$$

the vertices of the parallelohedron R. One will determine with the help of equations

$$\sum\sum a_{ij}l_{ir}^{(k)}l_{jr}^{(k)} + 2\sum \alpha_{ik}l_{ir}^{(k)} = 0. \quad (r = 1, 2, \ldots, n, k = 1, 2, \ldots, s)$$

Each vertex (α_{ik}) $(k = 1, 2, \ldots, s)$ is characterised by n systems of integers

$$(l_{i1}^{(k)}), (l_{i2}^{(k)}), \ldots, (l_{in}^{(k)}), \quad (k = 1, 2, \ldots, s) \tag{1}$$

the determinant $\pm\omega_k$ of which does not cancel each other out.
Let us indicate, to make short, n systems (1) by a symbol

$$\{l_{ir}^{(k)}\}.$$

All the vertices of the primitive parallelohedron R will be characterised by a set of symbols

$$\{l_{ir}^{(1)}\}, \{l_{ir}^{(2)}\}, \ldots, \{l_{ir}^{(s)}\}. \tag{2}$$

This declared, let us examine another primitive parallelohedron R' corresponding to another positive quadratic form $\sum\sum a'_{ij}x_ix_j$. It can turn out that all the vertices of the parallelohedron R' will alsobe characterised by the symbols (2). One will say in this case that the two parallelohedra R and R' belong to the same type.
Definition. One will call the various parallelohedra all the vertices of which are characterised by the set of symbols (2), "belonging to the same type."

57 One can characterise a type of primitive parallelohedra in many ways.
Let us consider a set (R) of congruent primitive parallelohedra which corresponds to a positive quadratic form $\sum\sum a_{ij}x_ix_j$.
All the vertices of parallelohedra belonging to the set (R) can be divided into classes of congruent vertices. Let us indicate by τ the number of incongruent vertices belonging to the various classes.
Any vertex of a primitive parallelohedron is congruent to n vertices of parallelohedron, this results in that

$$S = (n+1)\tau.$$

Let (α_i) be any one vertex of parallelohedra of the set (R). One will define it with the help of $n+1$ equations

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} = A. \; (k = 0, 1, 2, \ldots, n) \tag{3}$$

The $n+1$ systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in})$$

characterise $n+1$ parallelohedra of the set (R) which are contiguous by the vertex (α_i). By indicating with (l_i) a system of arbitrary integers, one will characterise by the systems

$$(l_{i0}+l_i), (l_{i1}+l_i), \dots, (l_{in}+l_i) \tag{4}$$

all the congruent vertices of parallelohedra of the set (R).

By attributing to the variables $l_1, l_2, \dots, l_n$ any arbitrary values, one will characterise by $n+1$ systems (4) a class of congruent vertices.

One concludes this that a type of primitive parallelohedra can be characterised by τ systems (4).

58 To have more convenience in the notations, let us introduce in our studies the linear functions

$$u = \sum_{i=1}^{n} l_i x_i, \text{ and } u_k = \sum_{i=1}^{n} l_{ik} x_i. \quad (k = 0, 1, 2, \dots, n)$$

One will say that the symbol $(u_0, u_1, \dots, u_n)$ characterise the vertex (α_i) determined by the equations (3); the symbol $(u_0+u, u_1+u, \dots, u_n+u)$, u being a linear function in arbitrary integer coefficients, characterise a vertex congruent to the vertex (α_i).

Let us suppose that one had characterised by the symbols

$$(u_0^{(k)}, u_1^{(k)}, \dots, u_n^{(k)}), \quad (k = 1, 2, \dots, \tau) \tag{5}$$

τ congruent vertices of primitive parallelohedra belonging to the set (R). One will say that the set of symbols (5) characterise a type of primitive parallelohedra.

59 Let us examine the faces in various dimensions of primitive parallelohedra belonging to the same type.

Let $P(\nu)$ be a face in ν dimensions $(\nu = 0, 1, 2, \dots, n-1)$ of parallelohedra of the set (R) defined by the equations

$$\sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{ik} = \sum\sum a_{ij} l_{i0} + 2\sum \alpha_i l_{i0}. (k = 1, 2, \dots, n-\nu)$$

One will characterise this face by $n+1-\nu$ systems

$$(l_{i0}), (l_{i1}), \dots, (l_{i,n-\nu})$$

or by $n+1-\nu$ corresponding linear functions.

$$u_0, u_1, \dots, u_{n-\nu}.$$

All the faces in ν dimensions of parallelohedra of the set (R) which are congruent to the face $P(\nu)$ will be characterised by the systems

$$(l_{i0} l_i), (l_{i1} l_i), \dots, (l_{i,n-\nu} l_i)$$

or by the corresponding linear functions

$$u_0 + u, u_1 + u, \dots, u_{n-\nu} + u.$$

By making, for example, $l_i = -l_{i0}$ one obtains $n-\nu$ systems

$$(l_{i1} - l_{i0}), (l_{i2} - l_{i0}), \dots, (l_{i,n-\nu} - l_{i0}) \tag{6}$$

which enjoy the following property: all the determinants of the order $(n-\nu)^2$ which one can form from $n-\nu$ systems (6) do not cancel one another at the same time. Let us indicate by $\omega^{(n-\nu)}$ the greatest common divisor of these determinants. By declaring

$$x_i = \sum_{k=1}^{n-\nu} (l_{ik} - l_{i0}) \xi_k, \tag{7}$$

one will present a system (x_i) of integers by the linear forms where $\xi_1, \xi_2, \dots, \xi_{n-\nu}$ are integer or rational numbers which belong to $\omega^{(n-\nu)}$ sets

$$\xi_k = \vartheta_{kr} + y_k \quad (k = 1, 2, \dots, n-\nu; r = 1, 2, \dots, \omega^{(n-\nu)}) \tag{8}$$

where $y_1, y_2, \dots, y_{n-\nu}$ are arbitrary integers. Among the sets (8) is found a set where $\vartheta_{kr} = 0$, $k = 1, 2, \dots, n-\nu$ and which is composed of integer values of $\xi_1, \xi_2, \dots, \xi_{n-\nu}$.

In the case $\omega^{(n-\nu)} = 1$, the equalities (7) are possible only on condition that the number $\xi_1, \xi_2, \dots, \xi_{n-\nu}$ be integer.

The set (8) play an important role in the subsequent studies.

Let us indicate by the symbol $\sigma_{n-\nu}$ the number of incongruent faces in ν dimensions of primitive parallelohedra belonging to the type examined. By indicating with the symbol S_ν the number of faces in ν dimensions of corresponding primitive parallelohedron, one will have a formula

$$S_\nu = (n+1-\nu)\sigma_{n-\nu}. \quad (\nu = 0, 1, 2, \dots, n-1) \tag{9}$$

Definition of the set (L) of simplexes characterising a type of primitive parallelohedra

60 Let us suppose that $n+1$ systems

$$(l_{i0}), (l_{i1}), \dots, (l_{in}) \tag{1}$$

characterise a vertex of primitive parallelohedra belonging to the type examined.

Definition I. One will call correlative to the vertex of primitive parallelohedra characterised by the systems (1) *a simplex L having $n+1$ vertices*

$$(l_{i0}), (l_{i1}), \dots, (l_{in}).$$

The simplex L presents a set of points determined by the equalities

$$x_i = \sum_{k=0}^{n} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n} \vartheta_k = 1 \text{ and } \vartheta_k \geq 0. \quad (k = 0, 1, 2, \dots, n)$$

Let us indicate by (L) the set of simplexes correlative to the various vertices of a set (R) of primitive parallelohedra belonging to the type examined.

Definition II. One will say that a type of primitive parallelohedra is characterised by the set (L) of simplexes.

One will call congruent two simplexes characterised by the vertices

$$(l_{i0}), (l_{i1}), \dots, (l_{in}) \text{ and } (l_{i0}+l_i), (l_{i1}+l_i), \dots, (l_{in}+l_i),$$

$l_1, l_2, \dots, l_n$ being arbitrary integers.

All the simplexes of the set (L) can be divided into classes of congruent simplexes; the number of classes is expressed by the symbol σ_n defined by the formula (9) of the previous number.

With the help of equations

$$x_i = \sum_{k=0}^{n-\nu} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n-\nu} \text{ and } \vartheta_k \geq 0, \quad (k = 0, 1, 2, \dots, n-\nu)$$

one will determine a face in $n-\nu$ dimensions of the simplex L which is correlative to the face in ν dimensions of parallelohedra characterised by the systems

$$(l_{i0}), (l_{i1}), \dots, (l_{i[,]n-\nu}).$$

One concludes that the number of incongruent faces in $n-\nu$ dimensions of the set (L) of simplexes is expressed by the symbol $\sigma_{n-\nu}$ $(\nu = 0, 1, 2, \dots, n-1)$.

As all the vertices of simplexes of the set (L) are congruent, one will declare $\sigma_0 = 1$, and the formula (9) of Number 59

$$S_\nu = (n+1-\nu)\sigma_{n-\nu}$$

will hold for the values of $\nu = 0, 1, 2, \dots, n$, provided that one would admit $S_n = 1$.

61

Theorem I. The set (L) of simplexes uniformly fills the space in n dimensions.

Let us suppose that a point (x_i) be interior to a face of the simplex L characterised by the systems

$$(l_{i0}), (l_{i1}), \dots, (l_{i\nu}). \tag{2}$$

One will have

$$x_i = \sum_{k=0}^{\nu} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{\nu} \vartheta_k = 1 \text{ and } \vartheta_k > 0. \quad (k = 0, 1, 2, \dots, \nu) \tag{3}$$

Let us suppose that the point (x_i) be interior to a face in ν' dimensions of another simplex L' characterised by the vertices

$$(l'_{i0}), (l'_{i1}), \dots, (l'_{i\nu}).$$

One can write

$$x_i = \sum_{k=0}^{\nu'} \vartheta'_k l'_{ik} \text{ where } \sum_{k=0}^{\nu'} \vartheta'_k = 1 \text{ and } \vartheta'_k > 0.\ a; (k = 0, 1, 2, \dots, \nu') \tag{4}$$

Let $\sum\sum a_{ij}x_ix_j$ be a positive quadratic form which defines a set (R) of primitive parallelohedra belonging to the type examined.

Let us indicate by (α_i) and (α'_i) two vertices of parallelohedra of the set (R) which are correlative to the simplexes L and L'. One will have the equalities

$$\begin{cases} \sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} = A, \\ \sum\sum a_{ij}l'_{ik}l'_{jk} + 2\sum \alpha'_i l'_{ik} = A'. \end{cases} \quad (k = 0, 1, 2, \dots, n) \tag{5}$$

By putting down

$$\begin{cases} \sum\sum a_{ij}l'_{ik}l'_{jk} + 2\sum \alpha_i l'_{ik} = A + \rho_k, \\ \sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha'_i l_{ik} = A' + \rho'_k, \end{cases} \quad (k = 0, 1, 2, \dots, n) \tag{6}$$

one will have the inequalities

$$\rho_k \geq 0 \text{ and } \rho'_k \geq 0. \quad (k = 0, 1, 2, \dots, n)$$

From equalities (5) and (6) one derives

$$\begin{cases} A' - A + 2\sum(\alpha_i - \alpha_i')l_{ik}' = \rho_k, \\ A - A' + 2\sum(\alpha_i' - \alpha_i)l_{ik} = \rho_k'. \end{cases} \qquad (k = 0, 1, 2, \ldots, n)$$

By virtue of equalities (3) and (4), one obtains,

$$A' - A + 2\sum(\alpha_i - \alpha_i')x_i = \sum_{k=0}^{\nu'} \rho_k \vartheta_k',$$

$$A - A' + 2\sum(\alpha_i' - \alpha_i)x_i = \sum_{i=0}^{\nu} \rho_k' \vartheta_k.$$

By making the sum of these equalities, one finds

$$\sum_{k=0}^{\nu'} \rho_k \vartheta_k' + \sum_{k=0}^{\nu} \rho_k' \vartheta_k = 0.$$

It follows, because of (3) and (4), that

$$\rho_k = 0, \quad (k = 0, 1, 2, \ldots, \nu') \quad \text{and} \quad \rho_k' = 0. \quad (k = 0, 1, 2, \ldots, nu)$$

let us notice that the equality $\rho_k = 0$ is possible only on the condition that the system (l_{ik}') is found among the vertices of the simplex L, similarly, the equality $\rho_k' = 0$ is possible only on condition that the system (l_{ik}) is found among the vertices of the simplex L'.

One concludes that the systems

$$(l_{i0}'), (l_{i1}'), \ldots, (l_{i\nu'}') \tag{7}$$

characterise a face of the simplex L and that the systems (2) characterise a face of the simplex L'. As a point (x_i) can not be interior to two different faces of the same simplex, it results in that the systems (2) and (7) coincide; therefore the two simplexes L and L' are contiguous through the faces in ν dimensions characterised by the systems (2).

It remains to demonstrate that any point (x_i) of the space in n dimensions belongs to at least one simplex of the set (L).

To demonstrate this, let us take any one point (ξ_i) which is interior to the simplex L and draw any one curve C which joins the points (ξ_i) and (x_i). I say that all the points of the curve C will be situated in the simplexes

$$L, L', \ldots, L^{(m)}$$

belonging to the set (L). In effect, let us suppose that the point (x_i) not belong to the simplex L. The curve C will go beyond in one point (ξ_i') the boundary of the simplex L and will pass through a simplex L' which is contiguous to the simplex L through a face in any one number of dimensions and so on and so forth.

62

Theorem II. A point (x_i) the elements $x_1, x_2, \ldots, x_n$ of which are integers can be only one vertex of simplexes of the set (L).

Let us notice that there exist simplexes of the set (L) which passes the vertex (0); the number of these simplexes is expressed by the symbol S_0.

By effecting the translations of these simplexes the length of the vector $[x_i]$, one will obtain S_0 simplexes which possess the vertex (x_i). By virtue of Theorem I, the point (x_i) can not belong to other simplexes of the set (L).

Corollary. Suppose that a point (x_i) the elements $x_1, x_2, \ldots, x_n$ of which are integers, is not found among the vertices

$$(l_{i0}), (l_{i1}), \ldots, (l_{in})$$

of a simplex L. By writing

$$x_i = \sum_{i=0}^{n} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n} \vartheta_k = 1,$$

one will have among the numbers $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ at least one negative number.

Properties of symbols S_ν and σ_ν $(\nu = 0, 1, 2, \ldots, n)$.

63

Let us take any one positive integer m and consider a set K of points which are congruent to m^n points

$$\frac{g_1}{m}, \frac{g_2}{m}, \ldots, \frac{g_n}{m}$$

which one obtains by attributing to the numbers $g_1, g_2, \ldots, g_n$ the integer values verifying the inequalities

$$0 \leq g_k < m. \quad (k = 1, 2, \ldots, n)$$

Let us take any one point $(\frac{x_i}{m})$ of the set K and suppose that the point $(\frac{x_i}{m})$ be interior to any one face $P(\nu)$ of simplexes of the set (L) $(\nu = 0, 1, 2, \ldots, n)$. By virtue of Theorem I of number 61, all points of the set K which are interior to the face $P(\nu)$ can not be congruent.

Let us indicate by

$$P_k^{(\nu)}, \quad (k = 1, 2, \dots, \sigma_\nu; \nu = 0, 1, 2, \dots, n)$$

the various incongruent faces of simplexes of the set (L) and by the symbol

$$m_k^{(\nu)} \quad (k = 1, 2, \dots, \sigma_\nu; \nu = 0, 1, 2, \dots, n)$$

let us indicate the number of points of the set K which are interior to the face $P_k(\nu)$. ONe will have a formula

$$\sum_{\nu=0}^{n} \sum_{k=1}^{\sigma_\nu} m_k^{(\nu)} = m^n. \tag{1}$$

64 It is easy to determine the value of the symbol $m_k^{(\nu)}$. Let us indicate by

$$l_{i0}^{(k)}, l_{i1}^{(k)}, \dots, l_{i\nu}^{(k)}$$

the vertices of the face $P_k(\nu)$ and let us write

$$\frac{x_i}{m} = \sum_{r=0}^{\nu} \vartheta_r l_{ir}^{(k)} \text{ where } \sum \vartheta_r = 1 \text{ and } \vartheta_r > 0. \quad (r = 0, 1, 2, \dots, \nu)$$

These equalities can be written

$$\frac{x_i}{m} - l_{i0}^{(k)} = \sum_{r=1}^{\nu} \vartheta_r (l_{ir}^{(k)} - l_{i0}^{(k)}).$$

By indicating, to make short,

$$x_i - m l_{i0}^{(k)} = t_i, \ \ l_{ir}^{(k)} - l_{i0}^{(k)} = p_{ir}, \quad (r = 1, 2, \dots, \nu)$$

and

$$m\vartheta_r = \tau_r, \quad (r = 1, 2, \dots, \nu)$$

one will have

$$t_i = \sum_{r=1}^{\nu} \tau_r p_{ir} \text{ where } \sum_{r=1}^{\nu} \tau_r < m \text{ and } \tau_r > 0. \ (r = 1, 2, \dots, \nu) \tag{2}$$

Let us indicate by $\omega_k^{(\nu)}$ the greatest common divisor of determinants of the order ν^2 which one can form from ν systems

$$(p_{i1}), (p_{i2}), \dots, (p_{i\nu})$$

and suppose that the forms (2) represent the integers $t_1, t_2, \dots, t_n$, provided that the numbers $\tau_1, \tau_2, \dots, \tau_\nu$ belong to one of $\omega_k^{(\nu)}$ sets

$$\tau_r = \xi_{rh} + y_r \text{ where } r = 1, 2, \dots, \nu, h = 1, 2, \dots, \omega_k^{(\nu)}, \tag{3}$$

$y_1, y_2, \dots, y_\nu$ being arbitrary integers.

One can suppose that

$$0 < \xi_{rh} \le 1. \quad (r = 1, 2, \dots, \nu, h = 1, 2, \dots, \omega_k^{(\nu)})$$

By substituting the expressions of τ_r $(r = 1, 2, \dots, \nu)$ derived from equalities (3) in the inequalities (2), one obtains

$$\xi_{rh} + y_r > 0, (r = 1, 2, \dots, \nu) \quad \sum_{r=1}^{\nu} (\xi_{rh} + y_r) < m. \tag{4}$$

Let us indicate

$$\sum_{r=1}^{\nu} \xi_{rh} = a_h + \xi_h \tag{5}$$

where the integer a_h is determined after the conditions

$$0 \le \xi_p < 1. \tag{6}$$

The inequalities (4) can be replaced by the following ones:

$$y_r \ge 0, \ (r = 1, 2, \dots, \nu) \text{ and } \sum_{r=1}^{\nu} y_r \le m - a_h - 1.$$

The number of systems $(y_1, y_2, \dots, y_\nu)$ of integers $y_1, y_2, \dots, y_\nu$ verifying these inequalities is equal to

$$\frac{(m - a_h)(m + 1 - a_h) \cdots (m + \nu - 1 - a_h)}{1 \cdot 2 \cdots \nu.}$$

By replacing with $a_{hk}^{(\nu)}$ the number a_h corresponding to the various sets (3), one obtains the formula

$$m_k^{(\nu)} = \sum_{h=1}^{\omega_k^{(\nu)}} \frac{(m - a_{hk}^{(\nu)})(m + 1 - a_{hk}^{(\nu)}) \cdots (m + \nu - 1 - a_{hk}^{(\nu)})}{1 \cdot 2 \cdots \nu}. \tag{7}$$

By substituting in the equality (1), one finds

$$\sum_{\nu=0}^{n} \sum_{k=1}^{\sigma_\nu} \sum_{h=1}^{\omega_k^{(\nu)}} \frac{(m - a_{hk}^{(\nu)}) \cdots (m + \nu - 1 - a_{hk}^{(\nu)})}{1 \cdot 2 \cdots \nu} = m^n. \tag{8}$$

The formula obtained holds, whatever may be the positive integer value of m. One concludes this that this formula presents an identity.

65

By comparing the coefficients of m^n in the formula (8), one finds

$$\sum_{k=1}^{\sigma_n} \omega_k^n = n!.$$

It follows that

$$\sigma_n \leq n!.$$

Let us introduce in our studies the finite difference of different orders by defining them by the formula

$$\Delta^{(\mu)} f(m) = \sum_{k=0}^{\mu} (-1)^{\mu - k} \frac{\mu!}{k!(\mu - k)!} f(m + \mu).$$

The formula (8) gives

$$\sum_{\nu=\mu}^{n} \sum_{k=1}^{\sigma_\nu} \sum_{h=1}^{\omega_k^{(\nu)}} \frac{(m + \mu - a_{hk}^{(\nu)}) \cdots (m + \nu - 1 - a_{hk}^{(\nu)})}{1 \cdot 2 \cdots (\nu - \mu)} = \Delta^{(\mu)}(m^n). \qquad (\mu = 0, 1, 2, \ldots, n)$$

By making $m = 1$ in this formula and by noticing that

$$\frac{(\mu + 1 - a_{hk}^{(\nu)}) \cdots (\nu - a_{hk}^{(\nu)})}{1 \cdot 2 \cdots (\nu - \mu)} \geq 0$$

since, because of (5) and (6)

$$a_{hk}^{(\nu)} \leq \nu,$$

one finds

$$\sum_{k=1}^{\sigma_\mu} \omega_k^{(\mu)} \leq \Delta^{(\mu)}(m^n)_{m=1}. \; (\mu = 0, 1, 2, \ldots, n) \tag{9}$$

It follows that

$$\sigma_\mu \leq \Delta^{(\mu)}(m^n)_{m=1}. \; (\mu = 0, 1, 2, \ldots, n)$$

We have seen in Number 60 that

$$S_\nu = (n + 1 - \nu)\sigma_{n-\nu}, \; (\nu = 0, 1, 2, \ldots, n) \tag{10}$$

therefore

$$S_\nu \leq (n + 1 - \nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \; (\nu = 0, 1, 2, \ldots, n)$$

66

Let us examine the conditions which have to be fulfilled for the symbols S_ν $(\nu = 0, 1, 2, \ldots, n)$ to be expressed by the formula

$$S_\nu = (n + 1 - \nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \; (\nu = 0, 1, 2, \ldots, n) \tag{11}$$

By virtue of inequalities (9), it is necessary that

$$\omega_k^{(\mu)} = 1. \; (k = 1, 2, \ldots, \sigma_\mu; \mu = 0, 1, 2, \ldots, n) \tag{12}$$

These are the conditions necessary and sufficient for the formula (11) to hold. In effect, in the case $\omega_k^{(\nu)} = 1$, the formula (7) becomes

$$m_k^{(\nu)} = \frac{(m - \nu)(m + 1 - \nu) \cdots (m - 1)}{1 \cdot 2 \cdots \nu},$$

and the equality (8) takes the form

$$\sum_{\nu=0}^{n} \sigma_\nu \frac{(m - \nu) \cdots (m - 1)}{1 \cdot 2 \cdots \nu} = m^n.$$

It follows that

$$\sum_{\nu=\mu}^{n} \sigma_\nu \frac{(m+\mu-\nu)\cdots(m-1)}{1\cdot 2\cdots(\nu-\mu)} = \Delta^{(\mu)}(m^n),$$

and by making $m = 1$, one obtains

$$\sigma_\mu = \Delta^{(\mu)}(m^n)_{m=1}. \quad (\mu = 0, 1, 2, \dots, n)$$

It results in, because of (10), the formula (11).
Let us notice that the conditions (12) come down to a single condition

$$\sigma_n = n!.$$

We will see that there exists primitive parallelohedra which satisfy this condition.

67

Theorem. The faces in 1, 2, 3 and 4 dimensions of simplexes of the set (L) *enjoy the property that*

$$\omega_k^{(}\nu) = 1. \quad (k = 1, 2, \dots, \sigma_\nu; \nu = 1, 2, 3, 4)$$

The demonstration of the theorem introduced does not present difficulties.
Corollary. The number of faces in different dimensions of primitive parallelohedra in the space of 2, 3 and 4 dimensions is expressed by the formula (11).

1. By making in the formula (11) $n = 2$, one obtains

$$S_0 = 6 \text{ and } S_1 = 6.$$

2. By making in the formula (11) $n = 3$, one obtains

$$S_0 = 24, \quad S_1 = 36, \quad S_2 = 14.$$

3. By making in the formula (11) $n = 4$, one obtains

$$S_0 = 120, \quad S_1 = 240, \quad S_2 = 150, \quad S_3 = 30.$$

By studying the primitive parallelohedra in the space of 5 dimensions, I have come across parallelohedra the number of faces of which is not expressed by the formula (11).

68

We have seen that, in the case $\omega_k^{(\nu)} = 1$, one has

$$a_{kh}^{(\nu)} = \nu.$$

It is easy to demonstrate that, in the case $\omega_k^{(\nu)} > 1$, one will have this equality for a single set (3) which is composed of integers; for all the sets which remain, one will have the inequalities

$$2 \le a_{hk}^{(\nu)} \le \nu - 2. \quad (\nu \ge 5) \tag{13}$$

Let us make in the formula (8) $m = 0$. By noticing that

$$\frac{(-a_{hk}^{(\nu)})(1-a_{hk}^{(\nu)})\cdots(\nu-1-a_{hk}^{(\nu)})}{1\cdot 2\cdots\nu} = 0$$

so long as $a_{hk}^{(\nu)} \neq \nu$, one finds

$$\sum_{\nu=0}^{n} (-1)^\nu \sigma_\nu = 0.$$

By making in the formula (8) $m = -1$, one obtains, because of (13),

$$\sum_{\nu=0}^{n} (-1)^\nu (\nu+1)\sigma_\nu = (-1)^n.$$

By substituting in this formula the expression of σ_ν derived from the formula (10), one will have

$$\sum_{\nu=0}^{n} (-1)^\nu S_\nu = 1. \tag{14}$$

Let us notice that the equality obtained expresses a property of faces in different dimensions of primitive parallelohedra which is common to all the convex polyhedra of the space in n dimensions. † By making in the formula (14) $n = 3$, one will have

$$S_0 - S_1 + S_2 - S_3 = 1,$$

and as $S_3 = 1$, this becomes

$$S_0 + S_2 = 2 + S_1.$$

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† See: *Poincaré*, Sur la généralisation d'un théorème d'Euler relatif aux polyèdres. [On the generalisation of the theorem of Euler relative to the polyhedra] (Comptes Rendus des Séances de l'Académie de Paris, V. 117, p. 144)

This is the well known formula of Euler. ‡

Regulators and characteristics of edges of primitive parallelohedra.

69 Let us examine the set (R) of primitive parallelohedra belonging to a type of parallelohedra characterised by a set (L) of simplexes.

Let (α_i) be a vertex of parallelohedra of the set (R) determined by the equations

$$\sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{ik} = A. \quad (k = 1, 2, \ldots, n) \tag{1}$$

The system L correlative to the vertex (α_i) is characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}). \tag{2}$$

Let us indicate by $(\alpha_{ik}$ $(k = 0, 1, 2, \ldots, n))$ the vertices adjacent to the vertex (α_i) (Number 18). The simplex L_k $(k = 0, 1, 2, \ldots, n)$ correlative to the vertex (α_{ik}) will be characterised by the systems which one obtains from system (2) by replacing the vertex (l_{ik}) of the simplex L by a corresponding vertex (l'_{ik}) of the simplex L_k. The two simplexes L and L_k are contiguous by a face in $n-1$ dimensions P_k $(k = 0, 1, 2, \ldots, n)$ which is characterised by the systems

$$(l_{ih}). \quad (k = 0, 1, 2, \ldots, n; h \neq k)$$

The face P_k of simplexes L and L_k is correlative to an edge $[\alpha_i, \alpha_{ik}]$ of parallelohedra of the set (R).

Let us indicate by

$$\sum p_{ik} x_i = \delta_k, \quad (k = 0, 1, 2, \ldots, n)$$

the equation of the face P_k. As one has

$$\sum p_{ik} l_{ik} = \delta_k, \quad (h = 0, 1, 2, \ldots, n; h \neq k)$$

it becomes

$$\sum p_{ik}(l_{ik} - l_{ir}) = 0. \quad (h = 0, 1, 2, \ldots, n; r = 0, 1, 2, \ldots, n; h \neq k; r \neq k) \tag{3}$$

The equalities obtained define the number $p_{1k}, p_{2k}, \ldots, p_{nk}$ to a common factor close by. By supposing that $p_{1k}, p_{2k}, \ldots, p_{nk}$ be integer not having common divisor, one will determine by the equality (3) two systems (p_{ik}) and $(-p_{ik})$. One will call characteristic of the edge $[\alpha, \alpha_{ik}]$ or of the correlative face P_k one of the two systems $\pm(p_{ik})$ likewise.

By noticing that

$$\sum p_{ik} l_{ik} \neq \delta_k,$$

one will attach, for more precision, a supplementary condition

$$\sum p_{ik} l_{ik} > \delta_k.$$

Definition. One will call characteristic of the face P_k with regard to the simplex L the system (p_{ik}) which is well defined by the conditions

$$\sum p_{ik} l_{ik} > \delta_k, \; \sum p_{ik} l_{ih} = \delta_k. \quad (h = 0, 1, 2, \ldots, n, h \neq k) \tag{4}$$

Let us notice that the characteristic of the face P_k with regard to the simplex L_k will be the system $(-p_{ik})$. In effect, one will have

$$\sum p_{ik} l'_{ik} \neq \delta_k.$$

Let us suppose that

$$\sum p_{ik} l'_{ik} > \delta_k.$$

In this case the two simplexes L and L_k would be situated on the same side of the face P_k, and one could find a point interior to the simplex L which would be interior to the simplex L_k too, this is contrary to Theorem I demonstrated in Number (6). It is therefore necessary that

$$\sum p_{ik} l'_{ik} < \delta_k,$$

and the system $(-p_{ik})$ presents the characteristic of the face P_k with regard to the simplex L_k.

70 One will determine the vertex (α_{ik}) $(k = 0, 1, 2, \ldots, n)$ correlative to the simplex L_k by the equations

$$\sum\sum a_{ij} l_{ih} l_{jh} + 2\sum \alpha_{ik} l_{ih} = A_k \quad (h = 0, 1, 2, \ldots, n; h \neq k) \tag{5}$$

and

$$\sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha_{ik} l'_{ik} = A_k. \tag{6}$$

‡ Euler, Elementa doctrinae Solidorum. (Novi Comment. Petrop. 1758.)

From the equalities (1) and (5), one derive

$$2\sum(\alpha_{ik}-\alpha_i)l_{ih}=A_k-A.\ \ (h=0,1,2,\ldots,n;h\neq k) \tag{7}$$

As a result because of (3), one will have

$$\alpha_{ik}-\alpha_i=p_{ik}\rho_k.\ \ (i=1,2,\ldots,n;k=0,1,2,\ldots,n) \tag{8}$$

On the ground of the supposition made, the vertices (α_i) and (α_{ik}) $(k=0,1,2,\ldots,n)$ of primitive parallelohedra of the set (R) are simple.

It follows that,

$$\sum\sum a_{ij}l'_{ik}l'_{jk}+2\sum\alpha_{ik}l'_{ik}>A$$

and

$$\sum\sum a_{ij}l_{ik}l_{i[\text{sic}]k}+2\sum\alpha_{ik}l_{ik}>A_k.$$

By virtue of (1) and (6), one obtains

$$2\sum(\alpha_{ik}-\alpha_i)l_{ik}>A_k-A\ \text{ and }\ 2\sum(\alpha_{ik}-\alpha_i)l'_{ik}<A_k-A$$

and, because of (8), it becomes

$$2\rho_k\sum p_{ik}l_{ik}>A_k-A\ \text{ and }\ 2\rho_k\sum p_{ik}l'_{ik}<A_k-A. \tag{9}$$

As by virtue of (7) and (8), one has

$$2\rho_k\sum p_{ik}l_{ih}=A_k-A,\ \ (h=0,1,2,\ldots,n;h\neq k) \tag{10}$$

the inequalities (9) can be written

$$2\rho_k\sum p_{ik}(l_{ik}-l_{ih})>0,\ 2\rho_k\sum p_{ik}(l'_{ik}-l_{ih})<0.\qquad (h=0,1,2,\ldots,n;h\neq k) \tag{11}$$

By noticing that because of (4)

$$\sum p_{ik}(l_i-l_{ih})>0,\ \ (h=0,1,2,\ldots,n;h\neq k) \tag{12}$$

one finds

$$\rho_k>0,\ \ (k=0,1,2,\ldots,n) \tag{13}$$

and the second inequality (11) gives

$$\sum p_{ik}(l'_{ik}-l_{ih})<0,\ \ (h=0,1,2,\ldots,n;h\neq k) \tag{14}$$

or differently, because of (4),

$$\sum p_{ik}l'_{ik}<\delta_k,\ \ (k=0,1,2,\ldots,n) \tag{15}$$

that which we have demonstrated by another method.

71 By substituting in (6) the expression of α_{ik} derived from the equality (8), one obtains

$$\sum\sum a_{ij}l'_{ik}l'_{jk}+2\sum\alpha_i l'_{ik}+2\rho_k\sum p_{ik}l'_{ik}=A_k.$$

One will present this equality, because of (1), under the form

$$\sum\sum a_{ij}l'_{ik}l'_{jk}+2\sum\alpha_i l'_{0k}-\sum\sum a_{ij}l_{ik}l_{jk}-2\sum\alpha_i l_{ik}= A_k-A-2\rho_k\sum p_{ik}l'_{ik},$$

and lastly, by virtue of (10),

$$2\rho_k\sum p_{ik}(l_{ih}-l'_{ik})=\sum\sum a_{ij}l'_{ik}l_{jk}+2\sum\alpha_i l'_{ik}-\sum\sum\alpha_{ij}l_{ik}l_{jk}-2\sum\alpha_i l_{ik} \tag{16}$$

where $h=0,1,2,\ldots,n,\ h\neq k,\ k=0,1,2,\ldots,n.$

Definition II. One will call regulator of the edge $[\alpha_i,\alpha_{ik}]$ or of the correlative face P_k the positive parameter ρ_k defined by the formulae (8) and (16).

Let us notice that on the ground of equalities (3) and (8) the congruent edges and the congruent correlative faces possess the same regulator and the same characteristic.

One can determine the regulator ρ_k by other formulae.

Let us write

$$l'_{ik}=\sum_{r=0}^{n}\vartheta_r^{(k)}l_{ir}\ \text{ where }\ \sum_{r=0}^{n}\vartheta_r^{(k)}=1.\ \ (k=1,2,\ldots,n) \tag{17}$$

On the grounds of equations (1) and (17), one obtains

$$\sum\sum a_{ij}l'_{ik}l'_{jk}+2\sum \alpha_i l'_{ik} - \sum\sum a_{ij}l_{ik}l_{jk} - 2\sum \alpha_i l_{ik} = \sum\sum a_{ij}l'_{ik}l'_{jk} - \sum_{r=0}^{n} \vartheta_r^{(k)} \sum\sum a_{ij}l_{ir}l_{jr}.$$

By substituting in the formula (16), one finds

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = \sum\sum a_{ij}l'_{ik}l'_{jk} - \sum_{r=0}^{n} \vartheta_r^{(k)} \sum\sum a_{ij}l_{ir}l_{jr} \tag{18}$$

where $h = 0,1,2,\ldots,n; h \neq k; k = 0,1,2,\ldots,n.$

The formula obtained makes visible an important property of the regulator ρ_k: *the regulator ρ_k is expressed by a linear function of coefficients of the quadratic form $\sum\sum a_{ij}x_ix_j$.* By writing

$$\rho_k = \sum\sum p_{ij}^{(k)} a_{ij}, \tag{19}$$

one will have the rational coefficients $p_{ij}^{(k)} = p_{ji}^{(k)},\ i = 1,2,\ldots,n, j = 1,2,\ldots,n.$

72 By virtue of the formula (19), the regulator ρ_k will be perfectly determined if one knows the corresponding coefficients $p_{ij}^{(k)}$ $(i = 1,2,\ldots,n; j = 1,2,\ldots,n)$.

As the coefficients a_{ij} $(i = 1,2,\ldots,n; j = 1,2,\ldots,n)$ of the quadratic form $\sum\sum a_{ij}x_ix_j$ do not play any role in the determination of coefficients $p_{ij}^{(k)}$ $(i = 1,2,\ldots,n; j = 1,2,\ldots,n)$ which depend only on the simplexes L and L_k, one can replace in the previous formula the coefficients a_{ij} by the coefficients x_ix_j $(i = 1,2,\ldots,n; j = 1,2,\ldots,n)$.

By introducing the linear functions, as we have done in Number 58,

$$u_r = \sum l_{ir}x_i,\ v_r = \sum l'_{ir}x_i, \quad (r = 0,1,2,\ldots,n)$$

let us indicate by

$$u_r^{(k)} \text{ and } v_r^{(k)} \quad (r = 0,1,2,\ldots,n)$$

the values of these functions which correspond to the values of variables $x_1, x_2, \ldots, x_n$

$$x_i = p_{ik}. \quad (k = 0,1,2,\ldots,n)$$

By virtue of (4), one will have

$$u_h^{(k)} = \delta_k. \quad (h = 0,1,2,\ldots,n; h \neq k, u_k^{(k)} > \delta_k)$$

By virtue of (15), one will have

$$v_k^{(k)} < \delta_k.$$

Let us notice that the numbers $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ defined by the equality (7) will be determined by the equalities

$$v_k = \sum_{r=0}^{n} \vartheta_r^{(k)} u_r \text{ where } \sum \vartheta_r^{(k)} = 1.$$

By replacing in the formula (18) the coefficients a_{ij} by the coefficients $x_ix_j,\ i = 1,2,\ldots,n, j = 1,2,\ldots,n,$ one obtaines

$$2\rho_k(\delta_k - v_k^{(k)}) = (v_k)^2 - \sum_{r=0}^{n} \vartheta_r^{(k)}(u_r)^2. \quad (k = 0,1,2,\ldots,n) \tag{20}$$

To return the formula obtained to the formula (19), it suffices to replace in the equality

$$\rho_k = \sum\sum p_{ij}^{(k)} x_ix_j$$

the coefficients x_ix_j by the coefficients $a_{ij},\ i = 1,2,\ldots,n, j = 1,2,\ldots,n.$

Fundamental transformation of the form

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i - \sum\sum a_{ij}l_il_j - 2\sum \alpha_i l_i$$

73 By keeping the previous notations, let us indicate

$$F_{(L)}(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i - A \tag{1}$$

where one has admitted

$$A = \sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik}. \quad (k = 0,1,2,\ldots,n) \tag{2}$$

By introducing the variables $\xi_0, \xi_1, \ldots, \xi_n$ after the conditions

$$x_i = \sum_{r=0}^{n} \xi_r l_{ir} \text{ where } \sum_{r=0}^{n} \xi_r = 1, \tag{3}$$

one will present the function $F_{(L)}\ (x_1, x_2, \ldots, x_n)$ under the following form:

$$F_{(L)}(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij} x_i x_j - \sum_{r=0}^{n} \xi_r \sum\sum a_{ij} l_{ir} l_{jr}. \tag{4}$$

One concludes that the function $F_{(L)}\ (x_1, x_2, \ldots, x_n)$ is linear with regard to the coefficients $a_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ of an arbitrary quadratic form $\sum\sum a_{ij} x_i x_j$.

By making in the formula (1) $x_i = l_{ik}$, one obtains, because of (2),

$$F_{(L)}(l_{1k}, l_{2k}, \ldots, l_{nk}) = 0. \quad (k = 0, 1, 2, \ldots, n)$$

The equalities obtained hold, whatever may be the values of $a_{ij}\ , i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$.

Let us indicate by L^0 a simplex congruent to the simplex L and characterised by the vertices

$$(l_{i0} + l_i), (l_{i1} + l_i), \ldots, (l_{in} + l_i),$$

$l_1, l_2, \ldots, l_n$ being arbitrary integers.

By noticing that because of (3)

$$x_i + l_i = \sum_{r=0}^{n} \xi_r (l_{ir} + l_i) \text{ where } \sum_{r=0}^{n} \xi_r = 1,$$

one will have an equality

$$F_{(L^0)}(x_1 + l_1, x_2 + l_2, \ldots, x_n + l_n) = \sum\sum a_{ij}(x_i + l_i)(x_j + l_j) - \sum_{r=0}^{n} \xi_r \sum\sum a_{ij}(l_{ir} + l_i)(l_{jr} + l_j)$$

and after the reductions, it becomes

$$F_{(L^0)}(x_1 + l_1, x_2 + l_2, \ldots, x_n + l_n) = \sum\sum a_{ij} x_i x_j - \sum_{r=0}^{n} \xi_r \sum\sum a_{ij} l_{ir} l_{jr},$$

therefore, because of (4) one will have

$$F_{(L^0)}(x_1 + l_1, x_2 + l_2, \ldots, x_n + l_n) = F_{(L)}(x_1, x_2, \ldots, x_n). \tag{5}$$

By virtue of the formula (18) of Number 71, one will determine the regulator ρ_k in the formula

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = F_{(L)}(l'_{1k}, l'_{2k}, \ldots, l'_{nk}). \ (k = 0, 1, 2, \ldots, n) \tag{7}$$

74

Let us indicate

$$F_{L_k}(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij} x_i x_j + 2\sum \alpha_{ik} x_i - A_k, \ (k = 0, 1, 2, \ldots, n)$$

L_k being a simplex contiguous to the simplex L by the face P_k $(k = 0, 1, 2, \ldots, n)$.

By substituting in this equality the expression of α_{ik} defined from the formula (8) of Number 70, one obtains

$$F_{L_k}(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij} x_i x_j + 2\sum \alpha_i x_i + 2\rho_k \sum p_{ik} x_i - A_k$$

and, because of (1), one will have

$$F_{L_k}(x_1, x_2, \ldots, x_n) = F_{(L)}(x_1, x_2, \ldots, x_n) + 2\rho_k \sum p_{ik} x_i + A - A_k.$$

By substituting in this equality the expression of $A - A_k$ given by the formula (10) of Number 70, one finds

$$F_{L_k}(x_1, x_2, \ldots, x_n) = F_{(L)}(x_1, x_2, \ldots, x_n) + 2\rho_r \sum p_{ik}(x_i - l_{ih}). \ (h \neq k)$$

This formula can be written

$$F_{(L)}(x_1, x_2, \ldots, x_n) = F_{L_k}(x_1, x_2, \ldots, x_n) + 2\rho_k \sum p_{ik}(l_{ih} - x_i). \qquad (h \neq k, k = 0, 1, 2, \ldots, n) \tag{*}$$

The formula (*) obtained is capable of numerous and important applications.

75

Let us suppose that $x_1, x_2, \ldots, x_n$ be arbitrary integers and that the point (x_i) is not found among the vertices

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}) \tag{7}$$

of the simplex L. By admitting

$$x_i = \sum_{r=0}^{n} \xi_r l_{ir} \text{ where } \sum_{r=0}^{n} \xi_r = 1, \tag{8}$$

one will have by virtue of Theorem II of Number 62 among the numbers $\xi_0, \xi_1, \ldots, \xi_n$ at least one negative number. Let us suppose to fix the ideas that

$$\xi_k < 0. \tag{9}$$

By noticing that because of (3) and of the formula (4) in Number 69 one has

$$\sum p_{ik}(l_{ih} - x_i) = \xi_k \sum p_{ik}(l_{ih} - l_{ik})$$

and that

$$\sum p_{ik}(l_{ih} - l_{ik}) < 0, \quad (h \neq k)$$

one obtains, because of (9),

$$\sum p_{ik}(l_{ih} - x_i) > 0.$$

One concludes that the coefficient of $2\rho_k$ in the formula $(^*)$ is an integer and positive in the case considered. In the same manner, one will examine the function $F_{L_k}(x_1, x_2, \ldots, x_n)$ and so on.

76 Let us suppose that one have examined the simplex,

$$L, L', L'', \ldots, L^{(m)} \tag{10}$$

successively contiguous by the faces in $n-1$ dimensions the regulator of which present the function

$$\rho_1, \rho_2, \ldots, \rho_m.$$

Let us suppose that by applying the formula $(^*)$ to the simplexes (10) one have obtained the equalities

$$\begin{aligned} F_L(x_1, x_2, \ldots, x_n) &= F_{L'}(x_1, x_2, \ldots, x_n) + 2h_1\rho_1 \text{ where } h_1 > 0, \\ F_{L'}(x_1, x_2, \ldots, x_n) &= F_{L''}(x_1, x_2, \ldots, x_n) + 2h_2\rho_2 \text{ where } h_2 > 0, \\ &\cdots \\ F_{L^{(m-1)}}(x_1, x_2, \ldots, x_n) &= F_{L^{(m)}}(x_1, x_2, \ldots, x_n) + 2h_m\rho_m \text{ where } h_m > 0. \end{aligned}$$

It follows that

$$F_L(x_1, x_2, \ldots, x_n) = 2\sum_{k=1}^{m} h_k\rho_k + F_{L^{(m)}}(x_1, x_2, \ldots, x_n). \tag{11}$$

The procedure shown can not be prolonged indefinitely and one will always arrive at a simplex $L^{(m)}$ among the vertices of which is found the point (x_i).

To demonstrate this, let us notice that the coefficients $a_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ of the quadratic form $\sum\sum a_{ij}x_ix_j$ in the formulae obtained are arbitrary.

Let us suppose that one have chosen the positive quadratic form $\sum\sum a_{ij}x_ix_j$ which defines a set (R) of primitive parallelohedra belonging to the type characterised by the set (L) of simplexes.

We have seen in Number 70 that one will have the inequalities

$$\rho_k > 0. \quad (k = 1, 2, \ldots, m)$$

By virtue of the definition of the function $F_L(x_1, x_2, \ldots, x_n)$, one will have an inequality

$$F_L(x_1, x_2, \ldots, x_n) > 0,$$

whatever the integer values of $x_1, x_2, \ldots, x_n$ may be, abstraction made from vertices (7) of the simplex L. It results in

$$F_{(L^{(m)})}(x_1, x_2, \ldots, x_n) \geq 0$$

and the formula (11), in the case considered, gives

$$F_L(x_1, x_2, \ldots, x_n) \geq 2\sum_{k=1}^{m} h_k\rho_k.$$

As the coefficients h_k $(k = 1, 2, \ldots, m)$ are of positive integers and the regulators ρ_k $(k = 1, 2, \ldots, m)$ belong to a series of regulators corresponding to the incongruent faces of simplexes of the set (L), one concludes that the number m can not be increased indefinitely. As a result the series (10) will be terminated by a simplex $L^{(m)}$ among the vertices of which is found the point (x_i).

It follows that one will have indentically

$$F_{(L^{(m)})}(x_1, x_2, \ldots, x_n) = 0,$$

and the formula (11) becomes

$$F_L(x_1, x_2, \ldots, x_n) = 2 \sum_{[k]=1}^{m} h_k \rho_k \text{ where } (k = 1, 2, \ldots, m) \tag{12}$$

Let us notice that the formula obtained presents an identity which holds, whatever the values of coefficients $a_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ may be, provided that the regulators ρ_k $(k = 1, 2, \ldots, m)$ are expressed by the formula (6).

77

Fundamental theorem. Let us suppose that the regulators ρ_k $(k = 1, 2, \ldots, \sigma)$ corresponding to the various incongruent faces in $n-1$ dimensions of simplexes belonging to the set (L) be determined by the equations

$$\rho_k = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}^{(k)} a_{ij}. \quad (k = 1, 2, \ldots, \sigma)$$

For a quadratic form $\sum \sum a_{ij} x_i x_j$ to define a set (R) primitive parallelohedra belonging to the type characterised by the set (L) of simplexes, it is necessary and sufficient that the inequalities

$$\rho_k = \sum \sum p_{ij}^{(k)} a_{ij} > 0, \quad (k = 1, 2, \ldots, \sigma)$$

hold.

We have seen in Number 70 that the inequalities

$$\rho_k > 0, \ (k = 1, 2, \ldots, \sigma) \tag{13}$$

present the necessary conditions. Let us suppose the coefficients of a quadratic form $\sum \sum a_{ij} x_i x_j$ verify the inequalities (13). By virtue of the formula (12), one will have the inequality

$$F_{(L)}(x_1, x_2, \ldots, x_n) > 0$$

so long as the point (x_i) the element s of which are integers is not found among the vertices of the simplex L. By virtue of the definition established, the simplex L is in this case correlative to a simplex vertex (α_i) of parallelohedra corresponding to the quadratic form examined $\sum \sum a_{ij} x_i x_j$.

The simplex L is chosen arbitrary in the set (L) of simplexes, therefore all the simplexes of the set (L) are correlative to the simple vertices of parallelohedra corresponding to the quadratic form $\sum \sum a_{ij} x_i x_j$.

I argue that these parallelohedra do not possess other vertices, it is that which one will verify without trouble.

78

Let us notice that any quadratic form $\sum \sum a_{ij} x_i x_j$ verifying the inequalities (13) is positive. To demonstrate this, let us examine a simplex L among the vertices in which is found the point (0). One will have in this case

$$F_{(L)}(x_1, x_2, \ldots, x_n) = \sum \sum a_{ij} x_i x_j + 2 \sum \alpha_i x_i,$$

and consequently

$$F_{(L)}(x_1, x_2, \ldots, x_n) + F_{(L)}(-x_1, -x_2, \ldots, -x_n) = 2 \sum \sum a_{ij} x_i x_j.$$

The two points (x_i) and $(-x_i)$ can not be the vertices of the simplex L, the point (0) being excluded. This results in

$$F_{(L)}(x_1, x_2, \ldots, x_n) + F_{(L)}(-x_1, -x_2, \ldots, -x_n) > 0,$$

therefore

$$\sum \sum a_{ij} x_i x_j > 0,$$

whatever may be the integer values of $x_1, x_2, \ldots, x_n$, the system $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ being excluded.

Definition of quadratic forms with the help of regulators and corresponding characteristics.

79

Let us take any one quadratic form $\sum \sum a_{ij} x_i x_j$ in arbitrary coefficients. Let us choose n numbers $x_1, x_2, \ldots, x_n$ which are subject to the only condition: *the equality*

$$h_1 x_2 + h_2 x_2 + \ldots + h_n x_n = 0$$

is impossible so long as the numbers $h_1, h_2, \ldots, h_n$ are integers.

Let us examine a vector g made up of points

$$\frac{l_1}{m} + u x_i \text{ where } 0 \leq u \leq 1,$$

$l_1, l_2, \ldots, l_n$ being arbitrary integers and m being any one positive integer.

The vector g will traverse a certain number of simplexes belonging to the set (L). Let us indicate by

$$L_0, L_1, \ldots, L_s \tag{1}$$

the simplexes of the set (L) which contain the various parts of the set g. On the ground of the supposition made, the simplexes (1) are well defined by the vector g and are successively contiguous by the faces in $n-1$ dimensions. In effect, two adjacent simplexes L_k and L_{k+1} of the series (1) possess a common point (ξ_{ik})

belonging to the vector g, therefore the simplexes L_k and L_{k+1} are contiguous by a face in any number of dimensions. Let us suppose that this face be characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{i\nu}). \tag{2}$$

As

$$\xi_{ik} = \frac{l_i}{m} + u_k x_i \text{ where } 0 < u_k < 1,$$

one will have

$$\frac{l_i}{m} + u_k x_i = \sum_{r=0}^{\nu} \vartheta_r l_{ir} \text{ where } \sum \vartheta_r = 1 \text{ and } \vartheta_r > 0. \; (r = 0, 1, 2, \ldots, \nu) \tag{3}$$

By supposing that $\nu < n - 1$, one wil determine with the help of these equalities a system (h_i) of integers verifying the equation

$$h_1 x_1 + h_2 x_2 + \ldots + h_n x_n = 0,$$

which is contrary to the hypothesis, therefore it is necessary that $\nu = n - 1$ and that the point (ξ_{ik}) be interior to a face in $n-1$ dimensions which is common to the simplexes L_k and L_{k+1}.

Let us suppose that $\nu = n - 1$. By indicating with (p_{ik}) the characteristic of the face (P_k) characterised by the systems (2) with regard to the simplex L_k, one will have, by virtue of the formula (4) of Number 69,

$$\sum p_{ik} l_{ir} = \delta_k, \quad (r = 0, 1, 2, \ldots, n-1)$$

and the equalities (3) give

$$\sum p_i (\frac{l_i}{m} + u_k x_i) = \delta_k$$

and consequently

$$u_k \sum p_{ik} x_i = \delta - \sum p_{ik} \frac{l_i}{m}.$$

As $\sum p_{ik} x_i \neq 0$, on the ground of the supposition made, the equality obtained defines a point (ξ_{ik}) of the vector g which is interior to the face P_k. This results in that the vector g does not possess other points common to the face P_k. By attributing to the parameter u a negative variation δu sufficiently small, one will define a point $\frac{l_i}{m} + (u_k + \delta u) x_i$ of the vector g which is interior to the simplex L_k. By attributing to the parameter u a variable $\delta u > 0$, one obtains a point $\frac{l_i}{m} + (u_k + \delta u) x_i$ which is interior to the simplex L_{k+1}.

As in these cases one has

$$\sum p_{ik} (\frac{l_i}{m} + (u_k + \delta u) x_i) > \delta_k, \quad (\delta u < 0)$$

and

$$\sum p_{ik} (\frac{l_i}{m} + (u_k + \delta u) x_i) < \delta_k, \quad (\delta u > 0)$$

it becomes

$$\sum p_{ik} x_i < 0. \tag{4}$$

By indicating with ρ_k the regulator of the face P_k with regard to the chosen quadratic form $\sum a_{ij} x_i k_j$ $(k = 0, 1, 2, \ldots, s-1)$, let us apply the formula $(^*)$ of Number 74 to the simplexes (1). One will have the equalities

$$F_{(L_0)}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) = F_{L_1}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n)$$
$$+ 2\rho_0 \sum p_{i0} (l_{ih_0} - \frac{l_i}{m} - x_i),$$
$$\ldots$$
$$F_{L_{s-1}}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) = F_{L_s}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n)$$
$$+ 2\rho_{s-1} \sum p_{i,s-1} (l_{ih_{s-1}} - \frac{l_i}{m} - x_i).$$

It follows that

$$F_{(L_0)}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) = 2 \sum_{k=0}^{s-1} \rho_k \sum i = 1^n p_{ik} (l_{ih_k} - \frac{l_i}{m} - x_i)$$
$$+ F_{L_s}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n). \tag{5}$$

Until now, the integers $l_1, l_2, \ldots, l_n$ had been arbitrary. Let us suppose that the integers $l_1, l_2, \ldots, l_n$ satisfy the conditions

$$0 \leq l_i < m. \; (i = 1, 2, \ldots, n) \tag{6}$$

Let us indicate by K the set of incongruent points $(\frac{l_i}{m})$ verifying these inequalities.

The number of points belonging to the set K is equal to m.

Let us apply the formula (5) to all the points of the set K and make the sum of equalities obtained. One will have a formula

$$F_{(L_0)}(\frac{l_1}{m}+x_1,\ldots,\frac{l_n}{m}+x_n)= \\ 2\sum \rho_k \sum p_{ik}(l_{ih_k}-\frac{l_i}{m}-x_i)+\sum F_{(L_s)}(\frac{l_1}{m}+x_i,\ldots,\frac{l_n}{m}+x_n). \tag{7}$$

All the sums which are formed in this formula can be determined with a certain approximation.

80 Let us suppose that the simplex L_0 be characterised by the systems

$$(l_{i0}),(l_{i1}),\ldots,(l_{in}).$$

On the ground of the supposition made, the point $\left(\frac{l_i}{m}\right)$ belongs to the simplex L_0. As there exist only a finite number of simplexes of the set L to which belong the points $\left(\frac{l_i}{m}\right)$ verifying the inequalities (6), one concludes that one can determine a positive parameter λ in such a manner that the inequalities

$$|l_{ik}|\le\lambda \ \ (i=1,2,\ldots,n;k=0,1,2,\ldots,n) \tag{8}$$

holds.

In this case, the corresponding value of the function

$$F_{(L_0)}\left(\frac{l_1}{m}+x_1,\frac{l_2}{m}+x_2,\ldots,\frac{l_n}{m}+x_n\right)$$

can be presented under the form

$$F_{(L_0)}\left(\frac{l_1}{m}+x_1,\ldots,\frac{l_n}{m}+x_n\right)=\sum\sum a_{ij}x_ix_j+\epsilon_0+\sum\epsilon_ix_i$$

where the coefficients $\epsilon_0,\epsilon_1,\ldots,\epsilon_n$ do not exceed in numerical value a fixed limit ϵ which does not depend on coefficients of the quadratic form $\sum\sum a_{ij}x_ix_j$ and on the choice of the set (L) of simplexes.

Let us examine the function $F_{(L_s)}\left(\frac{l_1}{m}+x_1,\ldots,\frac{l_n}{m}+x_n\right)$. After the proposition made, the point $\left(\frac{l_i}{m}+x_i\right)$ belongs to the simplex L_s. Let us determine the integers $t_1,t_2,\ldots,t_n$ after the conditions

$$0\le\frac{l_i}{m}+x_i+t_i<1,\ \ (i=1,2,\ldots,m) \tag{9}$$

and indicae by L'_s the simplex congruent to the simplex L_s which obtains by a translation of the simplex L_s the length of the vector $[t_i]$.

By virtue of the formula (5) of Number 73, one will have

$$F_{(L_s)}\left(\frac{l_1}{m}+x_1,\ldots,\frac{l_n}{m}+x_n\right)=F_{L'_s}\left(\frac{l_1}{m}+x_1+t_1,\ldots,\frac{l_n}{m}+x_n+t_n\right). \tag{10}$$

By indicating with

$$(l'_{i0}),\ldots,(l'_{in})$$

the vertices of the simplex L'_s, one will have, by virtue of (9), the inequalities (8):

$$|l'_{ik}|\le\lambda. \quad (i=1,2,\ldots,n;k=0,1,2,\ldots,n)$$

It follows that the numerical value of the function

$$F_{L'_s}\left(\frac{l_1}{m}+x_1+t_1,\ldots,\frac{l_n}{m}+x_n+t_n\right),$$

because of (8) and (9), does not exceed a fixed limit φ which depends only on coefficients of the quadratic form $\sum\sum a_{ij}x_ix_j$ and on the choice of the set (L) of simplexes. By virtue of (10), one can write

$$F_{(L_s)}\left(\frac{l_1}{m}+x_1,\ldots,\frac{l_n}{m}+x_n\right)=\varphi_0 \ \text{ where } \ |\varphi_0|\le\varphi.$$

By substituting in the formula (7) the results obtained, one will present it under the following form

$$m_n\left(\sum\sum a_{ij}x_ix_j+\epsilon_0+\sum\epsilon_ix_i\right)=2\sum\rho_k\sum_{i=1}^{n}p_{ik}\left(l_{ih}-\frac{l_i}{m}-x_i\right) \tag{11}$$

In this formula, the coefficients $\epsilon_0,\epsilon_1,\ldots,\epsilon_n$ do not exceed in numerical value a fixed limit which does not depend on numbers $x_1,x_2,\ldots,x_n$.

81 Let us determine the coefficients of $2\rho_k$ in the formula obtained.

To that effect, let us choose any one face P in $n-1$ dimensions of simplexes of the set (L). Let us suppose that the face P is characterised by the systems

$$(0),(l_{(i1)}),\ldots,(l_{(i,n-1)}).$$

Let us indicate by ρ the regulator and by $\pm(p_i)$ the characteristic of the face P with regard to the quadratic form $\sum\sum a_{ij}x_ix_j$. One will suppose, on the ground of (4), that

$$\sum p_i x_i < 0. \tag{12}$$

Let us suppose that the vector g is made up of points

$$\frac{l_i}{m} + ux_i \text{ where } 0 \le u \le 1$$

and corresponding to a point $\left(\frac{l_i}{m}\right)$ belonging to the set K possesses a point which is interior to a face P' congruent to the face P.

By supposing that the face P' is characterised by the systems

$$(g_i), (l_{i1} + g_i), \dots, (l_{i,n-1}, g_i),$$

one will have, on the ground of the supposition made,

$$\frac{l_i}{m} + ux_i = \sum_{k=0}^{n-1} \vartheta_k (g_i + l_{ik}) \text{ where } \sum_{k=0}^{n-1} \vartheta_k = 1 \text{ and } \vartheta_k > 0.$$

The corresponding value of the coefficient of 2ρ in the formula (11) is expressed by the sum

$$\sum\sum_{i=1}^{n} p_i (l_{ih} - \frac{l_i}{m} - x_i) \tag{14}$$

which extends to all the faces P' congruent to the face P verifying the equalities (13), provided that the points $\left(\frac{l_i}{m}\right)$ belong to the set K.

Let us indicate, to make short,

$$\sum_{i=1}^{n} p_i x_i = -\Delta \tag{15}$$

One will have, because of (12),

$$\Delta > 0.$$

As the system (l_{ih}) in the sum (14) indicate any one vertex of the face P', one can write down

$$l_{ih} = g_i,$$

and the equalities (13) and (15) give

$$\sum_{i=1}^{n} p_i \left(g_i - \frac{l_i}{m} - x_i \right) = (1-u)\Delta.$$

Therefore, the study of the coefficient of 2ρ in the formula (11) comes down to the evaluation of the sum

$$\sum (1-u)\Delta \text{ where } 0 < u < 1. \tag{16}$$

82 Let us designate, to summarise,

$$-mg_i + l_i = h_i. \tag{17}$$

The parameter u verifying the equalities (13) is expressed by the formula

$$u = \frac{1}{m\Delta} \sum p_i h_i, \tag{18}$$

and as $0 < u < 1$, it becomes

$$0 < \sum p_i h_i < m\Delta.$$

By indicating with τ the integer verifying the inequalities

$$0 < \tau < m\Delta, \tag{19}$$

let us write

$$\sum p_i h_i = \tau. \tag{20}$$

By virtue of (18), the corresponding value of the parameter u will be

$$u = \frac{\tau}{m\Delta}.$$

Let us substitute the expression found of the parameter u in the equalities (13), it will become, because of (17),

$$\frac{\tau}{\Delta} x_i + h_i = m \sum_{k=1}^{n-1} \vartheta_k l_{ik} \text{ where } \sum_{k=1}^{n-1} \vartheta_k < 1 \text{ and } \vartheta_k > 0. \tag{21}$$

$$(k = 1, 2, \ldots, n-1)$$

This stated, let us notice that one can attribute to the number τ an arbitrary value verifying the inequalities (19). For similar values of τ to exist, it is necessary that

$$m\Delta > 1,$$

Let us suppose that the positive integer m satisfies this condition. In this case, the finding of the sum (16) comes down to the solution of a sum

$$\sum (1-u)\Delta = \sum_{\tau>0}^{\tau<m\Delta} m_\tau (\Delta - \frac{\tau}{m}) \tag{22}$$

where m_τ indicates the number of systems (h_i) of integers $h_1, h_2, \ldots,\ h_n$ verifying the equalities (20) and (21).

83 It is easy to determine the number m_τ.

Let us indicate by (h_i^0) a system of integers verifying the equality

$$\sum p_i h_i^0 = 1. \tag{23}$$

As, on the ground of the supposition made, the integers $p_1, p_2, \ldots, p_n$ have no common divisor, the systems of integers verifying this equality always exist.

One will determine all the systems (h_i) of integers verifying the equality (20) with the help of formulae

$$h_i = \tau h_i^0 + \sum_{k=1}^{n-1} \tau_k l_{ik} \tag{24}$$

where the rational numbers $\tau_1, \tau_2, \ldots, \tau_{n-1}$ belong to certain sets

$$\tau_k = \xi_{kr} + y_k, \quad (k = 1, 2, \ldots, n-1; r = 1, 2, \ldots, \omega) \tag{25}$$

ω being the greatest commondivisor of n determinants of the order $(n-1)^2$ which one can form from $n-1$ systems

$$(l_{i1}), (l_{i2}), \ldots, (l_{i,n-1}).$$

By substituting the expressions of $h_1, h_2, \ldots, h_n$ derived from equalities (24) in the equalities (21), one obtains

$$\frac{\tau}{\Delta} x_i + \tau h_i^0 = \sum_{k-1}^{n-1} (m\vartheta_k - \tau_k) l_{ik}. \tag{26}$$

Let us notice that the numerical value of the determinant of n systems

$$(x_i), (l_{i1}), \ldots, (l_{i,n-1})$$

is expressed by the formula

$$\pm \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ l_{11} & l_{21} & \cdots & l_{n1} \\ & \cdots & \cdots & \\ l_{1,n-1} & l_{2,n-1} & \cdots & l_{n,n-1} \end{vmatrix} = -\sum x_i p_i \omega = \omega\Delta.$$

Let us indicate by $\lambda_{1k}, \lambda_{2k}, \ldots, \lambda_{nk},\ (k = 1, 2, \ldots, n-1)$ the minor determinants which are defined by the equalities

$$\sum_{i=1}^{n} \lambda_{ik} l_{ik} = \omega\Delta, \quad (k = 1, 2, \ldots, n-1)$$

$$\sum \lambda_{ik} x_i = 0, \ \sum \lambda_{ik} l_{ir} = 0. \quad (r = 1, 2, \ldots, n-1; r \neq k)$$

The equalities (26) give

$$\tau \sum \lambda_{ik} h_i^0 = \omega\Delta(m\vartheta_k - \tau_k), \quad (k = 1, 2, \ldots, n-1)$$

and as a result

$$m\vartheta_k = \tau_k + \frac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0. \quad (k = 1, 2, \ldots, n-1)$$

By virtue of (21) one obtains the inequalities

$$\tau_k + \frac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0 > 0, \quad (k = 1, 2, \ldots, n-1)$$

$$\sum_{k=1}^{n-1} (\tau_k + \frac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0) < m.$$

Considering the set (25), one finds

$$\begin{cases} y_k + \xi_{kr} + \dfrac{\tau}{\omega\Delta}\sum \lambda_{ik} h_i^0 > 0, \\ \displaystyle\sum_{k=1}^{n-1}\left(y_k\xi_{kr} + \frac{\tau}{\omega\Delta}\sum \lambda_{ik} h_i^0\right) < m. \end{cases} \quad (k = 1, 2, \ldots, n-1) \tag{27}$$

Let us write

$$y_k\xi_{kr} + \frac{\tau}{\omega\Delta}\sum \lambda_{ik} h_i^0 = y_k' + \nu_k \text{ where } 0 < \nu_k \leq 1 \quad (k = 1, 2, \ldots, n-1)$$

and

$$\sum_{k=1}^{n-1}\nu_k = a_r^{(\tau)} + \nu \text{ where } 0 \leq \nu < 1, \tag{28}$$

$y_1', \ldots, y_{n-1}'$ and $a_r^{(\tau)}$ being integers.
The inequalities (27) will be replaced by the following ones:

$$\sum_{k=1}^{n-1} y_k' < m - a_r^{(\tau)} - \nu, \; y_k' > -\nu_k, \quad (k = 1, 2, \ldots, n-1)$$

or differently

$$\sum_{k=1}^{n-1} y_k' \leq m - a_r^{(\tau)} - 1 \text{ and } y_k' \geq 0. \quad (k = 1, 2, \ldots, n-1)$$

The number of systems $(y_1', y_2', \ldots, y_{n-1}')$ of integers verifying these inequalities is equal to

$$\frac{(m - a_r^{(\tau)})(m + 1 - a_r^{(\tau)})\cdots(m + n - 2 - a_r^{(\tau)})}{1\cdot 2\cdots(n-1)}.$$

One concludes that the symbol m_τ which expresses the number of solutions of equations (20) and (21) in integers is equal to the sum

$$m_\tau = \sum_{\tau=1}^{\omega}\frac{(m - a_r^{(\tau)})\cdots(m + n - 2 - a_r^{(\tau)})}{1\cdot 2\cdots(n-1)}.$$

84

By substituting in the sum (26), one obtains

$$\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta - \frac{\tau}{m}\right)m_\tau = \sum_{\tau>0}^{\tau<m\Delta}\left(\Delta - \frac{\tau}{m}\right)\sum_{r=1}^{\omega}\frac{(m - a_r^{(\tau)})\cdots(m + n - 2 - a_r^{(\tau)})}{1\cdot 2\cdots(n-1)}. \tag{29}$$

Let us find a value approached by the sum obtained. By noticing that because of (28)

$$0 \leq a_r^{(\tau)} \leq n - 1,$$

one can write

$$\sum_{r=1}^{\omega}\frac{(m - a_r^{(\tau)})\cdots(m + n - 2 - a_r^{(\tau)})}{1\cdot 2\cdots(n-1)} = \frac{\omega}{(n-1)!}m^{n-1} + \delta_\tau m^{n-2}$$

where $|\delta_t au|$ does not exceed a fixed limit which doesnot depend on the number m.
By substituting in the sum (27), one finds

$$\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta - \frac{\tau}{m}\right)m_\tau = \frac{\omega}{(n-1)!}\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta - \frac{\tau}{m}\right) + \delta\Delta^2 m^{n-1}$$

where $|\delta|$ does not exceed a fixed limit which does not depend on $x_1, x_2, \ldots, x_n$ and on the number m.
By noticing that

$$\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta - \frac{\tau}{m}\right) = \Delta^2\frac{m}{2} - \frac{\Delta}{2} + \frac{\vartheta}{m} \text{ where } 0 \leq \vartheta < \frac{1}{8},$$

one can write

$$\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta - \frac{\tau}{m}\right)m_\tau = \frac{1}{2}\frac{\omega\Delta^2}{(n-1)!}m^n + m^{n-1}(\delta\Delta^2 + \delta'\Delta + \delta'') \tag{30}$$

where $\delta, \delta', \delta''$ do not exceed in numerical value a fixed limit.

By substituting in the formula (11) the coefficient found of 2ρ, one will have, because of (19),

$$m^n(\sum\sum a_{ij}x_ix_j + \epsilon_0 + \sum \epsilon_i x_i) = \frac{m^n}{(m-1)!}\sum_{k=1}^{\sigma}\rho_k\omega_k(\sum p_i x_i)^2 + m^{n-1}\sum_{k=1}^{\sigma}2\rho_k[\delta_k(\sum p_{ik}x_i)^2 + \delta'_k\sum p_{ik}x_i + \delta''_k]. \tag{31}$$

In the formula obtained the coefficients $\epsilon_0, \epsilon_1, \ldots, \epsilon_n, \delta_k, \delta'_k, \delta''_k$ $(k = 1, 2, \ldots, \sigma)$ do not exceed in numerical value a fixed limit which depend only on coefficients of the quadratic form $\sum\sum a_{ij}x_ix_j$ snf on the choice of the set (L) of simplexes.

Let us replace in the formula obtained the numbers $x_1, x_2, \ldots, x_n$ by the numbers $mx_1, mx_2, \ldots, mx_n$. As these numbers satisfy the conditions imposed on the numbers $x_1, x_2, \ldots, x_n$, the formula (31) is applicable and one obtains

$$m^n(m^2\sum\sum a_{ij}x_ix_j + \epsilon_0 + m\sum \epsilon_i x_i) = \frac{m^{n-2}}{(n-1)!}\sum_{k=1}^{\sigma}\rho_k\omega_k(\sum p_{ik}x_i)^2 + m^{n-1}\sum_{k=1}^{\sigma}2\rho_k[\delta_k m^2(\sum p_{ik}x_i)^2 + \delta'_k m\sum p_{ik}x_i + \delta''_k].$$

By dividing the two parts of the formula obtained by m^{n+2}, let us make the positive integer m increase indefinitely, it will become

$$\sum\sum a_{ij}x_ix_j = \frac{1}{(n-1)!}\sum_{k=1}^{\sigma}\rho_k\omega_k(p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{nk}x_n)^2. \tag{32}$$

The sum which is found in the second member of the formula obtained extends to all the incongruent faces in $n-1$ in $n-1$ dimensions of simplexes of the set (L).

We have deduced the formula (34) by supposing that the numbers $x_1, x_2, \ldots, x_n$ form a irreducible basis. As the two parts of the formula (32) present two quadratic forms, one concludes that the formula (32) present an indentity. This results in that the formula (32) can be written

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}a'_{ij} = \frac{1}{(n-1)!}\sum_{k=1}^{\sigma}\rho_k\omega_k\sum_{i=1}^{n}\sum_{j=1}^{n}a'_{ij}p_{ik}p_{jk} \tag{I}$$

where one has written

$$\rho_k = \sum_{i=1}^{n}\sum_{j=1}^{n}p_{ij}^{(k)}a_{ij}, \quad (k = 1, 2, \ldots, \sigma)$$

the two quadratic forms $\sum\sum a_{ij}x_ix_j$ and $\sum\sum a'_{ij}x_ix_j$ being arbitrary.

Section V.
Properties of the set (Δ) of quadratic forms corresponding to the various types of primitive parallelohedra.

Definition of the domain of quadratic form corresponding to a type of primitive parallelohedra.

85 Let us suppose that a type of primitive parallelohedra is characterised by a set (L) of simplexes.

Let us indicate by

$$\rho_k = \sum\sum p_{ij}^{(k)}a_{ij} \quad (k = 1, 2, \ldots, \sigma)$$

the regulators which correspond to the various incongruent faces in $n-1$ dimensions of simplexes of the set (L).

Definition. One will call domain of quadratic forms corresponding to the type of primitive parallelohedra characterised by the set (L) of simplexes a domain Δ in quadratic forms verifying the inequalities

$$\rho_k = \sum\sum p_{ij}^{(k)}a_{ij} \geq 0. \ (k = 1, 2, \ldots, \sigma) \tag{1}$$

On the ground of the fundamental theorem of Number 77, for a quadratic form f to define a set (R) of primitive parallelohedra belonging to the type characterised by the set (L) of simplexes, it is necessary and sufficient that the form f is interior to the domain Δ. This results in that the domain Δ is of $\frac{n(n+1)}{2}$ dimensions.

86 Among the inequalities (1) can be found dependent inequalities. Let us suppose that one has chosen a system of independent inequalities

$$\rho_1 \geq 0, \rho_2 \geq 0, \ldots, \rho_m \geq 0$$

which define the domain Δ. With the help of independent regulators $\rho_1, \rho_2, \ldots, \rho_m$ one will present all the regulators under the form

$$\rho_k = \sum_{r=1}^{m} h_r^{(k)} \rho_r \quad \text{where} \quad h_r^{(k)} \geq 0. \quad (r = 1, 2, \ldots, m; k = 1, 2, \ldots, \sigma) \tag{2}$$

Let us observe that any one quadratic form $\sum\sum a_{ij}x_ix_j$ does not verify the equations

$$\rho_1 = 0, \rho_2 = 0, \ldots, \rho_m = 0$$

because the equalities (2) give

$$\rho_k = 0, \quad (k = 1, 2, \ldots, \sigma)$$

and, by virtue of the formula (I) of Number 84, one has

$$\sum\sum a_{ij}a'_{ij} = 0,$$

$\sum\sum a'_{ij}x_ix_j$ being an arbitrary form; it follows that

$$a_{ij} = 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

To the domain Δ, therefore, the conclusion deduced in my first mémoire cited † are applicable.
Let us indicate by

$$\varphi_1, \varphi_2, \ldots, \varphi_s \tag{3}$$

the quadratic forms which characterise the various edges of the domain Δ.
The domain Δ of quadratic forms will be determined by the equalities

$$\sum\sum a_{ij}x_ix_j = \sum_{u=1}^{s} u_k\varphi_k \quad \text{where} \quad u_k \geq 0, \quad (k = 1, 2, \ldots, s)$$

$u_1, u_2, \ldots, u_s$ being positive arbitrary parameters or zeros.
Let us notice that by virtue of the formula (I) of Number (84), each form $\varphi_k (k = 1, 2, \ldots, s)$ of the series (3) will have for expression

$$\varphi_k = \sum_{r=1}^{\sigma} \lambda_r^{(k)} (p_{1r}x_1 + p_{2r}x_2 + \ldots + p_{nr}x_n)^2$$
$$\text{where} \quad \lambda_r^{(4)} \geq 0. \quad (r = 1, 2, \ldots, \sigma; k = 1, 2, \ldots, \sigma)$$

Properties of independent regulators

87 By keeping the notations from Number 69–74 let us suppose that a simplex L of the set (L) is characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}).$$

Let us suppose that among the regulators

$$\rho_0, \rho_1, \ldots, \rho_n$$

which correspond to the various faces in $n-1$ dimensions of the simplex L, is found at least one independent regulator. Let us suppose, to fix the ideas, that ρ_0 is a similar regulator.
Let us indicate by L_k $(k = 0, 1, 2, \ldots, n)$ the simplexes which are contiguous to the simplex (L) through the faces in $n-1$ dimensions characterised by the systems

$$(l_{ih}). \quad (h = 0, 1, 2, \ldots, n; h \neq k; k = 0, 1, 2, \ldots, n)$$

Let us suppose that by replacing in the simplex L the vertex (l_{ik}) by a vertex (l'_{ik}), one obtains the simplex L_k $(k = 0, 1, 2, \ldots, n)$.
By virtue of the formula (6) of Number 73, one will have

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = F_{(L)}(l'_{1k}, l'_{2k}, \ldots, l'_{nk}). \quad (k = 0, 1, 2, \ldots, n; h \neq k) \tag{1}$$

Let us admit

$$l'_{ik} = \sum_{k=0}^{n} \vartheta_r^{(k)} l_{ik} \quad \text{where} \quad \sum_{r=0}^{n} \vartheta_r^{(k)} = 1. \quad (k = 0, 1, 2, \ldots, n) \tag{2}$$

As, because of the inequality (14) of Number 70,

$$\sum p_{ik}(l_{ih} - l'_{ik}) > 0, \quad (h \neq k)$$

it becomes, by virtue of (2)

$$\vartheta_k^{(k)} < 0. \quad (k = 0, 1, 2, \ldots, n) \tag{3}$$

† This journal V. 133, p. 97

Let us examine the numbers
$$\vartheta_0^0, \vartheta_1^0, \ldots, \vartheta_n^0 \tag{4}$$
which correspond to the independent regulator ρ_0. One will have, because of (3),
$$\vartheta_0^0 = 0.$$

I say that among the numbers $\vartheta_1^0, \vartheta_2^0, \ldots, \vartheta_n^0$ at least two numbers are positive. As $\sum_{k=0}^{n} \vartheta_k^0 = 1$, it is evident that at least one number, for example ϑ_n^0, will be positive. Let us suppose that ϑ_n^0 is the only positive number in the series (4).

Let us indicate, to fix the ideas,
$$\vartheta_0^0 < 0, \vartheta_1^0, \ldots, \vartheta_\lambda^0 < 0, \vartheta_{\lambda[+1]}^0 = 0, \ldots, \vartheta_{n-1}^0 = 0, \vartheta_n^0 > 0. \tag{5}$$

The corresponding value of the function $F_{(L)}(l'_{10}, \ldots, l'_{n0})$, by virtue of the formula (4) of Number 78, can be presented under the form
$$\begin{aligned} F_{(L)}(l'_{10}, \ldots, l'_{n0}) &= \sum\sum a_{ij}(l'_{i0} - l_{in})(l'_{j0} - l_{jn}) \\ &- \sum_{k=0}^{\lambda} \vartheta_k \sum\sum a_{ij}(l_{ik} - l_{in})(l_{jk} - l_{jn}). \end{aligned} \tag{6}$$

By virtue of the formula (I) of Number 84 and of inequalities (5), one can present this equality under the form
$$F_{(L)}(l'_{10}, \ldots, l'_{n0}) = \sum_{r=1}^{\sigma} h_r \rho_r \quad \text{where} \quad h_r \geq 0,$$
and as on the other hand, because of (1)
$$2\rho_0 \sum p_{i0}(l_{ih} - l'_{i0}) = F_{(L)}(l'_{i0}, \ldots, l'_{n0}), \tag{7}$$
it becomes
$$\rho_0 = \sum_{r=1}^{\sigma} g_r \rho_r \quad \text{where} \quad g_r \geq 0. \quad (r = 1, 2, \ldots, \sigma)$$

We have supposed that ρ_0 is an independent regulator, therefore it is necessary that
$$g_2 = 0 \ \text{ so long as a regulator } \ \rho_r \ \text{ is not proportional to } \ \rho_0.$$

The formula (6) gives
$$\sum\sum a_{ij}(l'_{i0} - l_{in})(l'_{j0} - l_{jn}) = \delta\rho_0 \quad \text{where} \quad \delta > 0,$$
$$\sum\sum a_{ij}(l_{ik} - l_{in})(l_{jk} - l_{jn}) = \delta_k\rho_0 \quad \text{where} \quad \rho_k > 0.$$
$$(k = 0, 1, 2, \ldots, \lambda)$$

It follows that one has identically,
$$\sum\sum a_{ij}(l'_{i0} - l_{in})(l'_{j0} - l_{jn}) = \frac{\delta}{\delta_k} \sum\sum a_{ij}(l_{ik} - l_{in})(l_{jk} - l_{jn}).$$

For this identity to hold, it is necessary and sufficient that
$$l'_{i0} - l_{in} = \sqrt{\frac{\delta}{\delta_k}}(l_{ik} - l_{in}).$$

By virtue of Theorem I of Number 51, the numbers $l'_{i0} - l_{in}$ $(i = 1, 2, \ldots, n)$, and $l_{ik} - l_{in}$ $(i = 1, 2, \ldots, n)$, do not have common divisor, one concludes that
$$l'_{i0} = l_{ik},$$
which is impossible.

Let us indicae, to fix the ideas,
$$\vartheta_0^0 < 0, \vartheta_1^0 < 0, \ldots, \vartheta_\lambda^0 < 0, \vartheta_{\lambda+1}^0 = 0, \ldots, \vartheta_\mu^0 = 0, \vartheta_{\mu+1}^0 > 0, \ldots, \vartheta_n^0 > 0 \tag{8}$$
where $\lambda \geq 0$ and $\mu \leq n - 2$.

88 *Theorem: By replacing in the simplex L the vertices (l_{ik}), $k = 0, 1, 2, \ldots, \lambda$, successively by the vertex (l'_{i0}) one will obtain the simplexes*
$$L_0, L_1, \ldots, L_\lambda \tag{9}$$
which are contiguous to the simplex L and one to one by the faces in $n-1$ dimensions the regulators of which are proportional to the regulator ρ_0.

Let us apply the formula $(*)$ of Number 74 to the simplex L and L_k $(k = 0, 1, 2, \ldots, \lambda)$; one will have

$$F_{(L)}(l'_{10}, \ldots, l'_{n0}) = F_{(L_k)}(l'_{10}, \ldots, l'_{n0}) + 2\rho_k \sum p_{ik}(l_{ih} - l'_{i0}).$$
$$(h \neq k; n = 0, 1, 2, \ldots, \lambda)$$

By virtue of (8), one obtains,

$$\sum p_{ik}(l_{ih} - l'_{i0}) > 0. \quad (h \neq k; k = 0, 1, 2, \ldots, \lambda)$$

In view of (8), one finds

$$F_{(L_k)}(l'_{10}, \ldots, l'_{n0}) = \delta_k \rho_0 \text{ where } \delta_k \geq 0 \quad (k = 0, 1, 2, \ldots, \lambda)$$

and

$$\rho_k = u_k \rho_0 \text{ where } u_k > 0. \; (k = 0, 1, 2, \ldots, \lambda) \tag{10}$$

On the grounds of (1) and (7), it becomes

$$F_{(L)}(l'_{1k}, \ldots, l'_{nk}) = \omega_k F_{(L)}(l'_{10}, \ldots, l'_{n0}) \text{ where } \omega_k > 0. \; (k = 0, 1, 2, \ldots, \lambda)$$

The equality obtained presents an identity with regard to the coefficients of the quadratic form $\sum\sum a_{ij}x_i x_j$. One derives, because of (2), the equalities

$$\left(\vartheta_i^{(k)}\right)^2 - \vartheta_i^{(k)} = \omega_k \left((\vartheta_i^0)^2 - \vartheta_i^0\right), \quad (i = 0, 1, 2, \ldots, n)$$
$$\vartheta_i^{(k)} \vartheta_j^{(k)} = \omega_k \vartheta_i^0 \vartheta_j^0. \quad (i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, n; i \neq j)$$

As $\sum_{i=0}^n \vartheta_i^{(k)} = 1$ and $\sum_{i=0}^n \vartheta_i^0 = 1$, it is necessary that $\omega_k = 1$ and

$$\vartheta_i^{(k)} = \vartheta_i^0, \quad (i = 0, 1, 2, \ldots, n)$$

therefore

$$l'_{ik} = l'_{i0}. \quad (k = 0, 1, 2, \ldots, \lambda)$$

The formula (1) becomes in this case

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{i0}) = F_{(L)}(l'_{10}, \ldots, l'_{n0}). \quad (h \neq k; k = 0, 1, 2, \ldots, \lambda)$$

Let us notice that the simplexes

$$L, L_0, L_1, \ldots, L_\lambda$$

make up a group of perfectly determined simplexes corresponding to the independent regulator ρ_0. That which we have mentioned concerning the simplex L can be related back to all the simplexes of the series (9). All the simplexes which remain $L_{\lambda+1}, \ldots, L_n$ are contiguous to the simplex L through the faces the regulators of which are not proportional to ρ_0.

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Let us notice that the simplexes (9) make up a convex polyhedron K having $n + 2$ vertices

$$(l'_{i0}), (l_{i0}), \ldots, (l_{in}).$$

In effect, all the points of simplexes (9) belong to polyhedron K made up of points determined by the equalities

$$x_i = u l'_{i0} + \sum_{k=0}^n u_k l_{ik} \text{ where } u + \sum_{u=0}^n u_k = 1 \text{ and } u \geq 0, u_k \geq 0. \tag{11}$$
$$(k = 0, 1, 2, \ldots, n)$$

I argue that any point (x_i) determined by these equalities belongs to at least one of simplex (9).

Let us suppose, in the first place, that one has the inequalities

$$u_k + u\vartheta_k^0 \geq 0. \quad (k = 0, 1, 2, \ldots, n)$$

One will present the equation (11), because of (2), under the following form:

$$x_i = \sum_{k=0}^n (u_k + u\vartheta_k^0) l_{ik}$$

and, as $\sum_{k=0}^n (u_k + u\vartheta_k^0) = 1$, one concludes that the point (x_i) belongs to the simplex L.

This laid down, let us suppose that at least one of numbers $u_k + u\vartheta_k^0$, $k = 0, 1, 2, \ldots, n$, is negative. Let us choose among the numbers

$$\frac{u_0}{\vartheta_0^0}, \frac{u_1}{\vartheta_1^0}, \ldots, \frac{u_\lambda}{\vartheta_\lambda^0}$$

which are all negatives or zeros, because of (8) and (11), a number $\frac{u_k}{\vartheta^0_\lambda}$ the numerical value of which is the smallest. The point (x_i) determined by the equalities (11) belongs in this case to the simplex L_k. To demonstrate this, one will present the equalities (11) under the form

$$x_i = \left(u + \frac{u_k}{\vartheta_k}\right) l'_{i0} + \sum_r \left(u_r - u_k \frac{\vartheta_r}{\vartheta_k}\right) l_{ir}. \quad (r = 0, 1, 2, \ldots, n; r \neq k)$$

On the ground of suppositions made, one will have the inequalities

$$u + \frac{u_k}{\vartheta_k} > 0, \quad u_r - u_k \frac{\vartheta_r}{\vartheta_k} \geq 0, \quad (r = 0, 1, 2, \ldots, n; r \neq k)$$

and, as

$$u + \frac{u_k}{\vartheta_k} + \sum_r \left(u_r - u_k \frac{\vartheta_r}{\vartheta_k}\right) = 1,$$

one concludes that the point (x_i) belongs to the simplex (L_k) $(u = 0, 1, 2, \ldots, \lambda)$.

Let us examine the faces in $n-1$ dimensions of the polyhedron K. On the ground of conditions (8), the polyhedron K possesses $\mu - \lambda$ faces in $n-1$ dimensions Q_k which are characterised by $n+1$ vertices

$$(l'_{i0}), (l_{ih}). \quad (h = 0, 1, 2, \ldots, n; h \neq k; k = \lambda + 1, \ldots, \mu)$$

The vertex (l_{ik}) where $k = \lambda + 1, \ldots, \mu$ is opposite to the face $\theta_k (k = \lambda + 1, \ldots, \mu)$.

All the faces in $n-1$ dimensions of the polyhedron K which remain are characterised by n vertices. One will characterise them in the polyhedron K by two opposite vertices.

One obtains in this way $n - \mu$ faces P_k $(k = \mu + 1, \ldots, n)$ of the polyhedron K characterised by two opposite vertices (l'_{i0}) and l_{ik} $(k = \mu+1, \ldots, n)$ and one obtains $(\lambda+1)(n-\mu)$ faces $P_{kh} (h = 0, 1, 2, \ldots, \lambda; k = \mu + 1, \ldots, n)$ characterised by two opposite vertices l_{ik} and l_{ih}.

90 Let us notice that the polyhedron K is contiguous through the faces Q_k $(k = \lambda + 1, \ldots, \mu)$ to other independent regulator ρ_0.

to demonstrate this, let us examine the simplex L_k $(k = \lambda+1, \ldots, \mu)$ contiguous to the simplex L through the face in $n-1$ dimensions characterised by the vertices

$$(l_{ih}). \quad (h = 0, 1, 2, \ldots, n; h \neq k; k = \lambda + 1, \ldots, \mu)$$

This face presents a part of the corresponding face Q_k of the polyhedron K.

By applying the formula (*) of Number 74 to the simplexes L and L_k, one obtains

$$F_{(L)}(l'_{10}, \ldots, l'_{n0}) = F_{(L_k)}(l'_{10}, \ldots, l'_{n0}) + 2\rho_k \sum p_{ik}(l_{ih} - l'_{i0})$$
$$\text{where } h \neq k \text{ and } n = \lambda + 1, \ldots, \mu.$$

On the ground of conditions (8), one will have

$$\sum p_{ik}(l_{ih} - l'_{i0}) = 0, \quad (k = \lambda + 1, \ldots, \mu)$$

therefore, because of (7),

$$F_{(L_k)}(l'_{10}, \ldots, l'_{n0}) = F_{(L)}(l'_{10}, \ldots, l'_{n0}) = 2\rho_0 \sum p_{i0}(l_{ih} - l'_{i0}).$$
$$(h \neq 0; k = \lambda + 1, \ldots, \mu)$$

As the point (l'_{10}) is not found among the vertices of the simplex L_k, it is necessary that among the regulators of faces of the simplex L_k are found, by virtue of the equation obtained, regulators which are proportional to ρ_0.

By noticing that

$$l'_{i0} = \sum \vartheta^0_h l_{ih} + \vartheta^0_k l'_{ik}, \text{ where } \sum \vartheta^0_h + \vartheta^0_k = 1$$
$$(h = 0, 1, 2, \ldots, n; h \neq k; k = \lambda + 1, \ldots, \mu)$$

since because of (8), $\vartheta^0_k = 0$ $(k = \lambda + 1, \ldots, \mu)$, one concludes, on the ground of the previous theorem, that by replacing in the simplex L_k the vertices (l_{ih}) $(h = 0, 1, 2, \ldots, \lambda)$ by the vertex (l'_{i0}) one will obtain a group of simplexes

$$L_k, L^{(0)}_k, L^{(1)}_k, \ldots, L^{(\lambda)}_k, \quad (k = \lambda + 1, \ldots, \mu) \tag{12}$$

which are contiguous one to one by faces in $n-1$ dimensions the regulators of which are proportional to ρ_0.

Let us indicate by K_h the convex polyhedron made up of simplexes (12). One obtains the polyhedron K_k by replacing in the polyhedron K the vertex (l_{ik}) by the vertex l'_{ik} $(k = \lambda + 1, \ldots, \mu)$. One concludes that the polyhedra K and K_k are contiguous through the face Q_k.

Let us examine other faces of the polyhedra K. The face P_k $(k = \mu + 1, \ldots, n)$ belongs to the simplex L which is contiguous through the face P_k to the simplex L_k. The regulator ρ_k $(k = \mu + 1, \ldots, n)$ of this face can not be proportional to the independent regulator ρ_0.

It may turn out that any one of regulators corresponding to the various faces of the simplex L_k is not proportional to the regulator ρ_0. In this case, the polyhedron K will not be contiguous through the face P_k to any one analogous polyhedron corresponding to the independent regulator ρ_0.

It may also turn out that among the regulators of faces of the simplex L_k are found regulators which are proportional to ρ_0; in this case, the simplex L_k belongs to a convex polyhedron K_k which is contiguous to K through the face P_k $(k = \mu + 1, \ldots, n)$.

In the same way, one will examine the analogous faces P_{hk} of the polyhedron K $(h = 0, 1, 2, \ldots, \lambda; k = \mu + 1, \ldots, n)$.

By applying the procedure shown to the various incongruent simplexes of the set (L), one will determine the incongruent convex polyhedron

$$K, K_1, \ldots, K_{\omega-1}$$

which are made up of corresponding groups of simplexes belonging to the set (L).

Reconstruction of the set (L) of simplexes by another set (L') of simplexes.

91 One can partition the convex polyhedra

$$K, K_1, \ldots, K_{\omega-1} \tag{1}$$

corresponding to an independent regulator ρ into new simplexes.

By keeping the previous notations, let us examine the convex polyhedron K made up of simplexes

$$L, L_0, \ldots, L_\lambda. \tag{2}$$

The polyhedron K possesses $n + 2$ vertices

$$(l'_i), (l_{i0}), \ldots, (l_{in}).$$

it Theorem. By replacing in the simplex L characterised by the vertices

$$(l_{i0}), (l_{i1}), \ldots, (l_{in})$$

a value (l_{ik}) by the vertex (l'_i) where $k = \mu + 1, \ldots, n$, one obtains $n - \mu$ simplexes

$$L_{\mu+1}, \ldots, L'_n \tag{3}$$

which also make up the polyhedron K. The simplexes obtained do not belong to the set (L) of simplexes.

Let us write, as we have done in Number 87,

$$l'_i = \sum_{k=0}^{n} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n} \vartheta_k = 1 \tag{4}$$

and

$$\vartheta_0 < 0, \vartheta_1 < 0, \ldots, \vartheta_\lambda < 0, \vartheta_{\lambda+1} = 0, \ldots, \vartheta_\mu = 0, \vartheta_{\mu+1} > 0, \ldots, \vartheta_n > 0. \tag{5}$$

It is clear that each point of simplex (3) belongs to the polyhedron K.

Let (x_i) be any one point of the polyhedron K determined with the help of equalities

$$x_i = ul'_i + \sum_{k=0}^{n} u_k l_{ik} \text{ where } u + \sum u_k = 1, u \geq 0, u_k \geq 0. \ (k = 0, 1, \ldots, n) \tag{6}$$

Let us choose among the numbers

$$\frac{u_{\mu+1}}{\vartheta_{\mu+1}}, \ldots, \frac{u_n}{\vartheta_n}$$

the one which is the smallest. Let us suppose, to fix the ideas, that

$$\frac{u_r}{\vartheta_r} \geq \frac{u_k}{\vartheta_k}. \quad (r = \mu + 1, \ldots, n)$$

I argue that the point (x_i) belongs to the simplex L'_k. In effect, the equality (6) can be speculated, because of (4), under the form

$$x_i = \left(u + \frac{u_k}{\vartheta_k}\right) l'_i + \sum_h \left(u_h - u_k \frac{\vartheta_h}{\vartheta_k}\right) l_{ih}. \quad (h = 0, 1, 2, \ldots, n; h \neq k)$$

By observing that

$$u + \frac{u_k}{\vartheta_k} \geq 0, \ u_h - u_k \frac{\vartheta_h}{\vartheta_k} \geq 0, \quad (h = 0, 1, 2, \ldots, n; h \neq k)$$

and that

$$u + \frac{u_k}{\vartheta_k} + \sum_h \left(u_k - u_k \frac{\vartheta_h}{\vartheta_k}\right) = 1 \quad (h = 0, 1, 2, \ldots, n; h \neq k)$$

one concludes that the point (x_i) belongs to the simplex L'_k $(k = \mu + 1, \ldots, n)$.

The simplexes (3) can not belong to the set (L), because this set, by virtue of Theorem I of Number 61, uniformly partition the space in n dimensions.

92 Let us suppose that one has replaced in the set (L) the group of simplexes (2) by the corresponding group (3). Let us suppose that one has effected this reconstruction of simplexes of the set (L) with regard to all

the polyhedra which are congruent to the polyhedra (1). One obtains in this way a new set (L') of simplexes which enjoy the following properties.

1. The set (L') of simplexes uniformly fills the space in n dimensions.

2. The set (L') can be divided into classes of congruent simplexes and the number of different classes is finite.

Let us find the regulators and the characteristics of faces in $n-1$ dimensions of simplexes belonging to the set (L').

Let L' and L'_0 be any two simplexes of the set (L') which are contiguous through a face P in $n-1$ dimensions. Suppose that the two simplexes L' and L'_0 also belong to the set (L'). In this case the regulator and the characteristic of the face P in the set (L') do not change.

Let us suppose that at least one of the simplexes examined does not belong to the set (L) of simplexes. This simplex will belong in this case to a polyhedron which is congruent to a polyhedron of the series (1). Let us suppose to fix the ideas that this is the polyhedron K.

By noticing that the simplex examined is found among the simplexes (3) let us choose one of these simplexes L'_k $(k=\mu+1,\dots,n)$ and examine the regulators and the characteristics of all these faces in $n-1$ dimensions.

By virtue of the definition established, the simplex L'_k is characterised by the vertices

$$(l_{i0}),\dots,(l_{i,k-1}),(l'_i),(l_{i,k+1}),\dots,(l_{in}).\quad (k=\mu+1,\dots,n)$$

Let us indicate by P_{hk} a face in $n-1$ dimensions of the simplex L'_k which is opposite to the vertex (l_{ih}) $(h=0,1,2,\dots;h\neq k)$. By P'_k let us indicate the faces of simplex L'_k which is opposite to the vertex l'_i.

All the faces in $n-1$ dimensions of the simplex L'_k can be divided into three groups:

1. $P_{0k},P_{1k},\dots,P_{\lambda k}$ and P_k;
2. $P_{\lambda+1,k},\dots,P_{\mu k}$;
3. $P_{\mu+1,k},\dots,P_{k-1,k},P_{k+1,k},\dots,P_{nk}$.

93 Let us find the regulators of faces of the simplex L'_k belonging to the first group.

Let us examine, in the first place, the face P_k. As the face P_k is characterised by the vertices

$$(l_{i0}),\dots,(l_{i,k-1}),(l_{i,k+1}),\dots,(l_{in}),$$

it presents a face of the polyhedron K.

In the set (L) the face P_k would belong to two simplexes L and L_k. Two cases to distinguish:

First case: the simplex L_k belongs to the set (L').

Let us indicate by ρ_k the regulator and by (P_{ik}) the characteristic of the face P_k in the set (L) with regard to the simplex L.

Let us indicate by ρ'_k the regulator of the face P_k in the set (L'). The characteristics of the face P_k in the set (L') with regard to the simplex L'_k will be (P_{ik}).

By virtue of the definition established in Number 73, one can declare

$$F_{(L_k)}(l'_1,\dots,l'_n)=2\rho'_k\sum(-p_{ik})(l_{ih}-l'_i).\quad (h\neq k)$$

By applying the formula (*) of Number 74 to the simplex L and L_k, one obtains

$$F_{(L)}(l'_1,\dots,l'_n)=F_{(L_k)}(l'_1,\dots,l'_n)+2\rho_k\sum p_{ik}(l_{ih}-l'_i).\quad (h\neq k) \tag{7}$$

It follows that

$$\rho'_k=\rho_k+\frac{F_{(L)}(l'_1,\dots,l'_n)}{2\sum p_{ik}(-l_{ih}+l'_i)}.\quad (k=\mu+1,\dots,n) \tag{8}$$

We have seen in Number 87 that the function $F_{(L)}(l'_1,\dots,l'_n)$ is proportional to the independent regulator ρ. As, by virtue of (5),

$$\sum p_{ik}(-l_{ih}+l'_i)>0,\quad (h\neq k;k=\lambda+1,\dots,\mu)$$

the formula (8) can be written

$$\rho'_k=\rho_k+\delta_k\rho \text{ where } \delta_k>0.$$

Second case: the simplex L_k does not belong to the set (L').

In this case the simplex L_k belongs to a convex polyhedron K_k and the face P_k will belong in the set (L') to two new simplexes.

The face P_k in the set (L') belongs to the simplex L'_k and to a simplex which one obtains by replacing the vertex l'_{ik} of the simplex L_k by a new vertex which one will indicate by (l^0_i). Let us indicae by L^0_k the simplex which one obtains by replacing in the simplex L_k the vertex l'_{ik} by the vertex (l^0_i).

By virtue of the definition established in Number 73, one will have

$$F_{(L'_k)}(l^0_1,\dots,l^0_n)=2\rho'_k\sum p_{ik}(l_{ih}-l^0_i).\quad (h\neq k)$$

By applying the fundamental formula $(*)$ of Number 74 to the simplexes L_k and L'_k which are contiguous through the face P_k, one obtains

$$F_{(L_k)}(l_1^0,\ldots,l_n^0) = F_{(L'_k)}(l_1^0,\ldots,l_n^0) + \sum(-p_{ik})(l_{ih}-l_i^0)\frac{F_{(L_k)}(l'_1,\ldots,l'_n)}{\sum(-p_{ik})(l_{ih}-l'_i)}$$

and, because of (7), it becomes

$$\rho'_k = \rho_k + \frac{F_{(L)}(l'_1,\ldots,l'_n)}{2\sum p_{ik}(-l_{ih}+l'_i)} + \frac{F_{(L_k)}(l_1^0,\ldots,l_n^0)}{2\sum p_{ik}(l_{ih}-l_i^0)}. \qquad (9)$$

On the ground of the supposition made, the functions

$$F_{(L)}(l'_1,\ldots,l'_n) \text{ and } F_{(L_k)}(l_1^0,\ldots,l_n^0)$$

are proportional to the independent regulator ρ. As

$$\sum p_{ik}(-l_{ih}+l'_i) > 0 \text{ and } \sum p_{ik}(l_{ih}-l_i^0) > 0, \quad (k=\mu+1,\ldots,n)$$

the formula (9) can be written

$$\rho'_k = \rho_k + \delta_k\rho \text{ where } \delta_k > 0. \quad (k=\mu+1,\ldots,n)$$

In the same way, one will examine the regulator of the face P_{hk}; $(h=0,1,2,\ldots,\lambda)$.

As the face P_{hk} belongs in the set (L) to the simplex L_h $(h=0,1,2,\ldots,\lambda)$, one concludes that by designating the simplex L_h with the simplex L one will return to one of the two previous cases.

94 Let us find the regulators of faces of the simplex L'_k belonging to the second group. Choose one face P_{hk} $(h=\lambda+1,\ldots,\mu; k=\mu+1,\ldots,n)$ in this group.

The face P_{hk} presents a part of the face Q_k of the polyhedron K.

We have indicated in Number 90 by K_h the polyhedron which is contiguous to the polyhedron K through the face Q_h. The polyhedron K_h posses $n+1$ vertices

$$(l'_i),(l_{i0}),\ldots,(l_{i,h-1}),(l'_{ih}),(l_{i,h+1}),\ldots,(l_{in}).$$

In the set (L'), the polyhedron K_h is partitioned into simplexes

$$L'_{\mu+1,h},\ldots,l'_{n,h}$$

which one obtains by replacing in the simplexes

$$L'_{\mu+1},\ldots,L'_n$$

the vertex (l_{ih}) by the vertex (l'_{ih}) $(h=\lambda+1,\ldots,\mu)$.

One concludes that the face P_{hk} belongs to the set (L') to two simplexes

$$L'_k \quad\text{and}\quad L'_{kh}.$$

Let us indicate by ρ'_{hk} the regulator corresponding to the face P_{hk} in the set (L'). Notice that the characteristic (P'_{ih}) will be the same for all the faces P_{hk} where $k=\mu+1,\ldots,n$ because these faces make up the face Q_k of the polyhedron K.

In the set (L), the face Q_h is partitioned into faces

$$P_h, P_{h0},\ldots,P_{h\lambda}$$

of simplexes $L, L_0,\ldots,L_\lambda$ which have the same characteristic (p_{ih}). One concludes that

$$p'_{ih} = p_{ih},$$

provided that the characteristic (p_{ih}) is chosen with regard to the simplexes (2).

By virtue of the definition established in Number 73, one can write

$$F_{(L_k)}(l'_{ih},\ldots,L'_{nh}) = 2\rho'_{hk}\sum p_{ih}(l_{ir}-l'_{ih}). \quad (r\neq h)$$

By applying the formula $(*)$ of Number 74 to the simplexes L and L'_k, one will have

$$F_{(L)}(l'_{ih},\ldots,L'_{nh}) = F_{(L'_k)}(l'_{ih},\ldots,L'_{nh}) + \sum p_{ik}(l_{ir}-l'_{ih})\frac{F_{(L)}(l'_1,\ldots,l'_n)}{\sum p_{ik}(l_{ir}-l'_i)}.$$

Besides, one has

$$F_{(L)}(l'_{ih},\ldots,L'_{nh}) = 2\rho_h\sum p_{ih}(l_{ir}-l'_{ih}), \quad (r\neq h)$$

and consequently

$$\rho'_{hk} = \rho_h + \frac{\sum p_{ik}(l_{ir}-l'_{ih})}{\sum p_{ih}(l_{ir}-l'_{ih})}\cdot\frac{F_{(L)}(l'_1,\ldots,l'_n)}{2\sum(p_{ik}-l_{ir}+l'_i)}.$$

One can thus write

$$\rho'_{hk} = \rho_h + \delta_{hk}\rho. \quad (h=\lambda+1,\ldots,\mu; k=\mu+1,\ldots,n)$$

In the formula obtained, the number δ_{hk} can be positive, negative or zero.

95

Let us find the regulators of faces of the simplex L'_k belonging to the third group. Let P_{hk} be a face belonging to this group, $h = \mu+1, \ldots, n,\ h \neq k,\ k = \mu+1, \ldots, n$. The face P_{hk} belongs in the set (L') to two simplexes L'_k and L'_h of the series (3). By replacing in the simplex L'_k the vertices (l_{ih}) by the vertex (l_{ik}), one obtains the simplex L'_h. This results in that by indicating with ρ_{hk} the regulator and with $(p_i^{(hk)})$ the characteristic of the face P_{hk} with respect to the simplex L'_k, one will have

$$F_{(L'_k)}(l_{1k}, \ldots, l_{nk}) = 2\rho'_{hk} \sum p_i^{(hk)}(l_{ir} - l_{ik}). \quad (r \neq h; r \neq k) \tag{10}$$

The equality (4) can be written

$$l_{ik} = \frac{1}{\vartheta_k} l'_i + \sum_r \left(-\frac{\vartheta_r}{\vartheta_k}\right) l_{ir}. \quad (r = 0, 1, 2, \ldots, n; r \neq k)$$

By noticing that

$$\frac{1}{\vartheta_k} + \sum_r \left(-\frac{\vartheta_r}{\vartheta_k}\right) = 1,$$

one will determine the value of the function $F_{(L'_k)}(l_{1k}, \ldots, l_{nk})$, by virtue of the formula (4) of Number 73, by the equality

$$F_{(L'_k)}(l_{1k}, \ldots, l_{nk}) = \sum\sum a_{ij} l_{ik} l_{jk} - \frac{1}{\vartheta_k}\sum\sum a_{ij} l'_i l'_j$$
$$+ \sum_r \frac{\vartheta_r}{\vartheta_k} \sum\sum a_{ij} l_{ir} l_{jr}. \quad (r = 0, 1, 2, \ldots, n; r \neq k)$$

By recalling that because of (4)

$$F_{(L)}(l'_1, \ldots, l'_n) = \sum\sum a_{ij} l'_i l'_j - \sum_{k=0}^{n} \vartheta_k \sum\sum a_{ij} l_{ik} l_{jk},$$

and by comparing the two equalities obtained, one finds

$$F_{(L'_k)}(l_{1k}, \ldots, l_{nk}) = -\frac{1}{\vartheta_k} F_{(L)}(l'_1, \ldots, l'_n).$$

By substituting in the formula (10) the expression found of the function $F_{(L'_k)}(l_{1k}, \ldots, l_{nk})$, one obtains

$$\rho'_{hk} = -\frac{1}{\vartheta_k} \cdot \frac{F_{(L)}(l'_1, \ldots, l'_n)}{2\sum p_i^{(hk)}(l_{ir} - l_{ik})}.$$
$$(h = \mu+1, \ldots, n; k = \mu+1, \ldots, n; h \neq k; r \neq k)$$

One concludes that by admitting

$$\rho'_{hk} = -\delta_{hk}\rho, \quad (h = \mu+1, \ldots, n; k = \mu+1, \ldots, n; h \neq k)$$

one will have $\delta_{hk} > 0$.

Algorithm for the study of domains of quadratic forms which are contiguous to a given domain through the faces in $\frac{n(n+1)}{2} - 1$ dimensions.

96

Let us suppose that a domain Δ of quadratic form corresponding to a type of primitive parallelohedra which is characterised by the set (L) of simplexes is defined by the independent inequalities

$$\rho_k \geq 0. \quad (k = 1, 2, \ldots, m)$$

Let us suppose that one of these regulators is proportional to an independent regulator ρ and construct the set (L) of simplexes in another set (L') with the help of the procedure shown in Number 91–92.

Let us indicate by

$$\rho_1, \rho_2, \ldots, \rho_\sigma$$

all the regulators of incongruent faces of simplexes belonging to the set (L) and indicate by

$$\rho'_1, \rho'_2, \ldots, \rho'_\tau$$

all the regulators of faces of simplexes belonging to the set (L').

We have seen in Number 93–95 that all these regulators can be presented under the form

$$\begin{cases} \text{either} \quad \rho'_k = -\delta_k \rho \quad \text{where } \delta_k > 0, \\ \text{or} \quad \rho'_k = \rho_k + \delta_{kh}\rho \end{cases} \tag{1}$$

so long as a regulator ρ'_k is not proportional to ρ.

Let us examine the domain D' of quadratic forms determined by the inequalities

$$\rho'_k \geq 0. \quad (k = 1, 2, \ldots, \tau) \tag{2}$$

I argue that these inequalities define a domain of quadratic forms in $\frac{n(n+1)}{2}$ dimensions. By supposing the contrary, one will find the parameters u_k $(k = 1, 2, \ldots, \tau)$ positive or zero which reduce into an identity the equality

$$\sum_{k=1}^{\tau} u_k \rho'_k = 0 \text{ where } u_k \geq 0. \ (k = 1, 2, \ldots, \tau) \tag{3}$$

By virtue of formulae (1), this identity can be written

$$\sum_{k=1}^{\sigma} v_k \rho_k \pm v\rho = 0 \text{ where } v_k \geq 0. \ (k = 1, 2, \ldots, \sigma)$$

As the regulator ρ is independent it is necessary that $v_k = 0$ as long as a regulator ρ_k is not proportional to ρ. This results in that within the identity (3) one also has $u_k = 0$ so long as a regulator ρ'_k is not proportional to ρ. By virtue of (1), the identity (3) takes the form

$$\sum u_k(-\delta_k \rho) = 0 \text{ where } u_k \geq 0 \text{ and } \delta_k > 0,$$

which is impossible.

The domain Δ' defined by the inequalities (2) corresponds to a new type of primitive parallelohedra characterised by the set (L') of simplexes.

Let us notice that by virtue of inequalities (1), any quadratic form which is interior to the face of the domain Δ determined by the equation

$$\rho = 0 \tag{4}$$

belongs to the domain Δ' and vice-versa. One concludes that the two domain Δ and Δ' are contiguous through the face in $\frac{n(n+1)}{2} - 1$ dimensions determined by the equation (4).

Set (Δ) of domains of quadratic forms corresponding to the different types of primitive parallelohedra.

97

With the help of the algorithm explained in the previous Number, one can determine the domains of quadratic forms

$$\Delta_1, \Delta_2, \ldots, \Delta_m \tag{1}$$

which are contiguous to the domain Δ by their faces in $\frac{n(n+1)}{2} - 1$ dimensions, then one will determine the domains which are contiguous to the domains (1) and so on.

Let us indicate by (Δ) the set composed of all the domains of quadratic terms which correspond to the various types of primitive parallelohedra.

Theorem I. The set (Δ) of domains of quadratic forms uniformly divides the set of all the positive quadratic forms in n variables.

Let $\varphi(x_1, x_2, \ldots, x_n)$ be an arbitrary positive quadratic form. Let us choose a form $\varphi_0(x_1, x_2, \ldots, x_n)$ which is interior to the domain Δ and let us examine a vector g made up of forms

$$f = \varphi_0 + u(\varphi - \varphi_0) \text{ where } 0 \leq u \leq 1. \tag{2}$$

By making the parameter u grow in a continuous manner in the interval $0 \leq u \leq 1$, one will determine a series of domains

$$\Delta, \Delta', \ldots, \Delta^{(k)} \tag{3}$$

which are successively contiguous through the faces in $\frac{n(n+1)}{2} - 1$ dimensions and which contain the various forms of the vector g.

I argue that the series of domains (3) will always be terminated by a domain to which belong the given quadratic form φ.

To demonstrate this, let us indicate by

$$(l_{i1}), (l_{i2}), \ldots, (l_{i\tau}) \text{ where } \tau = 2^n - 1 \tag{4}$$

the systems which characterise the faces in $n-1$ dimensions of primitive parallelohedra belonging to the type which corresponds to the domain Δ of quadratic form.

Let us indicate by the symbol $N(f)$ a sum

$$N(f) = \sum_{h=1}^{\tau} f(l_{1h}, \ldots, l_{nh})$$

of values of a form $f(x_1, x_2, \ldots, x_n)$ corresponding to the systems (4).

Let us indicate, similarly, by

$$(l_{i1}^{(k)}), (l_{i2}^{(k)}), \ldots, (l_{i\tau}^{(k)}) \text{ where } \tau = 2^n - 1 \tag{5}$$

the systems which characterise the faces in $n-1$ dimensions of primitive parallelohedra belonging to the type which corresponds to a domain $\Delta^{(k)}$ of the series (3) and declare

$$N^{(k)}(f) = \sum_{h=1}^{\tau} f(l_{1h}^{(k)}, \ldots, l_{nh}^{(k)}). \quad (k = 1, 2, \ldots)$$

We have seen in Number 95 that the systems (4) and (5) are congruent with respect to the modulus 2. By virtue of the theorem of Number 48, one will have an inequality

$$N(f) < N^{(k)}f, \quad (k = 1, 2, \ldots)$$

as long as a quadratic form f is interior to the domain Δ. This results in that the inequality

$$N(f) \leq N^{(k)}(f) \quad (k = 1, 2, \ldots)$$

holds providing that a form f belongs to the domain Δ.

This stated, let us notice that by virtue of the supposition made, the form φ_0 is interior to the domain Δ, therefore one will have the inequality

$$N(\varphi_0) < N^{(k)}(\varphi_0). \;\; (k = 1, 2, \ldots) \tag{6}$$

Let f be a form of the vector g which belongs to the domain $\Delta^{(k)}$ of te series (3).

One will have an inequality

$$N(f) \geq N^{(k)}(f). \tag{7}$$

By noticing that because of (2)

$$N(f) = (1-u)N(\varphi_0) + uN(\varphi),$$
$$N^{(k)}(f) = (1-u)N^{(k)}(\varphi_0) + uN^{(k)}(\varphi),$$

the inequality (7) can be written

$$u\left[N(\varphi) - N^{(k)}(\varphi)\right] \geq (1-u)\left[N^{(k)}(\varphi_0) - N(\varphi_0)\right].$$

As $0 < u \leq 1$, this inequality gives, because of (6),

$$N^{(k)}(\varphi) \leq N(\varphi).$$

The quadratic form φ being positive, there exist only a limited number of different systems (5) verifying this inequality. Besides, there exist only a limited number of domains of forms belonging to the set (Δ) which are characterised by the same systems (5). One concludes this that the series (3) will always be terminated by a domain to which belong the given quadratic form φ.

Let us notice that a quadratic form φ which is interior to a domain Δ does not belong to any other domain of the set (Δ), since the primitive parallelohedron corresponding to the quadratic form φ will belong to the type characterised by the domain (Δ) and can not belong to any other type of parallelohedra.

Suppose that a positive quadratic form φ is interior to a face P in a certain number of dimensions of the domain Δ. The set of all the quadratic forms belonging to the face P will be perfectly determined by a certain type of nonprimitive parallelohedra. One concludes that the form φ can not belong to the domains which are contiguous through the face P.

98

By effecting the various transformation of the set (Δ) of quadratic forms with the help of substitutions of integer coefficients and of the determinant which is equal to ± 1, one will do only the permutation of domains of the set (Δ).

One concludes that the set (Δ) of domains of forms can be divided into classes of domains composed of equivalent domains.

Theorem II. The number of various classes of domains belonging to the set (Δ) *is finite.*

Let us choose any one domain Δ of the set (Δ) and let φ be a form which is interior to the domain Δ. We have seen in Number 54 that the positive quadratic form can be transformed into another equivalent form φ' which enjoys the property that the system (4) corresponding to the form φ' are made up of integers which do not exceed in numerical value a fixed limit.

The form φ' is interior to a domain Δ' which is equivalent to the domain Δ.

As the domain Δ' is characterised by the systems of integers which do not exceed in numerical value a fixed limit, there exist only a limited number of identical domains in the set (Δ).

99

With the help of the algorithm introduced in Number 96, one can successively determine all the representatives

$$\Delta, \Delta_1, \ldots, \Delta_{\mu-1} \tag{8}$$

of different classes of domains belonging to the set (Δ).

The domains obtained enjoy the same property as the domains of quadratic forms which have been studied in my first mémoire cited. † It results in that the domains (8) can serve in the reduction of positive quadratic forms. By calling reduced the positive quadratic forms which belong to the domains (8), one obtains a new reduction method of positive quadratic forms which is entirely analogous to a reduction method of positive quadratic forms introduced in the cited mémoire.

On the nonprimitive parallelohedra corresponding to positive quadratic forms.

100

Let us suppose that a positive quadratic form φ defines a primitive parallelohedron R.

By virtue of Theorem I of Number 97, the form φ belongs at least to the domain of the set (Δ). The form φ can not be interior to any one domain of the set (Δ) because otherwise the parallelohedron R would be primitive.

Therefore the form φ belongs to one face of domains of the set (Δ).

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† This Journal, V. 133

It results in that the coefficients of the form φ verify one of many linear equations

$$\sum\sum p_{ij}a_{ij} = 0$$

to rational coefficients p_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$

One concludes that a positive quadratic form $\sum\sum a_{ij}x_ix_j$ the coefficients of which present an irreducible basis can define only one primitive parallelohedron.

Let us suppose that the examined form φ is interior to a face P to any one number of dimensions of domains belonging to the set (Δ).

Let us indicate by

$$\Delta, \Delta', \ldots, \Delta^{(m)}$$

the domains of the set (Δ) which are contiguous through the face P.

By virtue of that which has been stated in Number 97, one will have the equalities

$$N(\varphi) = N'(\varphi) = \ldots = N^{(m)}(\varphi).$$

One concludes that a positive quadratic form φ can belong to only a finite number of domains of the set (Δ).

101

Let us suppose that an infinite series of quadratic forms

$$f_1, f_2, \ldots \tag{1}$$

is made up of forms which are interior to the domain Δ. Suppose that the forms of this series tend towards a limit φ.

The forms (1) define an infinite series of primitive parallelohedra

$$R_1, R_2, \ldots$$

belonging to the one type characterised by the domain Δ which tend towards a limit R.

One concludes that any nonprimitive parallelohedron R corresponding to a positive quadratic form φ can be considered as a limit of primitive parallelohedra (2).

Let us indicate by the symbol S_ν the number of faces in ν dimensions of the nonprimitive parallelohedron R and by S_ν^0 let us indicate the number of faces in ν dimensions of primitive parallelohedron (2) $(\nu = 0, 1, 2, \ldots, n-1)$.

As the faces of the nonprimitive parallelohedron R are made up of boundaries of faces of primitive parallelohedra belonging to the series (2), one concludes that

$$S_\nu \leq S_\nu^0. \quad (\nu = 0, 1, 2, \ldots, n-1)$$

We have seen in Number 65 that

$$S_\nu^0 \leq (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}, \quad (\nu = 0, 1, 2, \ldots, n)$$

and consequently

$$S_\nu \leq (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}, \quad (\nu = 0, 1, 2, \ldots, n)$$

Principal domain of the set (Δ).

102

Let us apply the general theory introduced in this mémoire to a positive quadratic form

$$f = nx_1^2 + nx_2^2 + \ldots + nx_n^2 - 2x_1x_2 - 2x_1x_3 \ldots - 2x_{n-1}x_n$$

where one has admitted

$$a_{11} = n \text{ and } a_{ij} = -1. \ (i \neq j; i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{1}$$

Let us find all the representations of the minimum of the form f in a set composed of all the systems of integers which are congruous to a system $(l_1, l_2, \ldots, l_n)$ with respect to the modulus 2. Let us admit

$$l_1 = 1, \ i = 1, 2, \ldots, \lambda \text{ and } l_i = 0, i = \lambda+1, \ldots, n. \ (\lambda = 1, 2, \ldots, n) \tag{2}$$

The problem described reduces to the study of the minimum of the form

$$f(l_1 + 2x_1, l_2 + 2x_2, \ldots, l_n + 2x_n)$$

in the set E composed of all the systems (x_i) of integers $x_1, x_2, \ldots, x_n$.

Let us notice that the form f, by virtue of equalities (1), can be written

$$f = \sum_{i=1}^{n} x_i^2 + \sum\sum_{i<j}(x_i - x_j)^2. \tag{3}$$

Each form

$$x_i^2, \ (i = 1, 2, \ldots, n) \ (x_i - x_j)^2, \ (i = 1, 2, \ldots, n; i < j; j = 1, 2, \ldots, n)$$

satisfied, by (2), the condition

$$\begin{cases} (l_i + 2x_i)^2 \geq l_i^2, & (i = 1, 2, \ldots, n) \\ (l_i - l_j + 2(x_i - x_j))^2 \geq (l_i - l_j)^2, & (i = 1, 2, \ldots, n; i < j; j = 1, 2, \ldots, n) \end{cases} \tag{4}$$

Whatever may be the integer values of $x_1, x_2, \ldots, x_n$. It follows, by (3), that

$$f(l_1 + 2x_1, l_2 + 2x_2, \ldots, l_n + 2x_n) \geq f(l_1, l_2, \ldots, l_n).$$

For the equality

$$\varphi(l_1 + 2x_1, l_2 + 2x_2, \ldots, l_n + 2x_n) = \varphi(l_1, l_2, \ldots, l_n)$$

to holds, it is necessary, by (3) and (4), that one had the equalities

$$(l_i + 2x_i)^2 = l_i^2, \;\; l_i - l_j + 2(x_i - x_j) = (l_i - l_j)^2. \qquad (i = 1, 2, \ldots, n; i < j; j = 1, 2, \ldots, n)$$

By virtue of (2), one obtains

$$x_i = 0 \;\text{ or }\; x_i = -l_i, \quad (i = 1, 2, \ldots, n)$$

therefore the form f possesses only two representations of the minimum $(l_1, l_2, \ldots, l_n)$ and $(-l_1, -l_2, \ldots, -l_n)$ in the set examined.

By attributing to the index λ in the inequalities (2) the values $\lambda = 1, 2 \ldots, n$ and by permuting the numbers $l_1, l_2, \ldots, l_n$, one obtains $2^n - 1$ systems which characterise, by virtue of the theorem of Number 48, the faces in $n-1$ dimensions of the parallelohedron R corresponding to the positive quadratic form f.

The parallelohedron R will be defined by $2(2^n - 1)$ independent inequalities

$$\begin{aligned}
&1 \cdot n \pm 2x_{k_1} \geq 0, \\
&2 \cdot (n-1) \pm 2(x_{k_1} + x_{k_2}) \geq 0, \quad (k_1 < k_2) \\
&\qquad \ldots \\
&\lambda(n - \lambda + 1) \pm 2(x_{k_1} + \ldots + x_{k_\lambda}) \geq 0, \qquad (k_1 < k_2 < \cdots < k_\lambda) \\
&\qquad \ldots \\
&n \cdot 1 \pm 2(x_{k_1} + x_{k_2} + \ldots + x_{k_n}) \geq 0, \qquad (k_1 < k_2 < \cdots < k_n)
\end{aligned}$$

where $k_1 = 1, 2, \ldots, n,\; k_2 = 2, \ldots, n, \ldots,\; k_\lambda = \lambda, \ldots, n,\; k_n = n.$

To have more convenience in the subsequent notations, let us write

$$u_0 = x_1 + x_2 \ldots + x_n,\; u_1 = -x_1,\; u_2 = -x_2, \ldots,\; u_n = -x_n \tag{5}$$

and notice that all the sums

$$\pm x_{k_1},\; \pm(x_{k_1} + x_{k_2}), \ldots,\; \pm(x_{k_1} + x_{k_2} + \ldots + x_{k_n})$$

are expressed by the sums

$$u_{h_0},\; u_{h_0} + u_{h_1}, \ldots,\; u_{h_0} + u_{h_1} + \ldots + u_{h_{n-1}}$$

where $h_0 < h_1 < h_2 < \cdots < h_{n-1}$ and $h_0 = 0, 1, 2, \ldots, n,\; h_1 = 1, 2, \ldots, n, \ldots,\; h_{n-1} = n-1,\; n.$

The inequalities which define the parallelohedron R can be written

$$\begin{cases}
1 \cdot n + 2u_{h_0} \geq 0, \\
2 \cdot (n-1) + 2(u_{h_0} + u_{h_1}) \geq 0, \; (h_0 < h_1) \\
\qquad \ldots \\
n \cdot 1 + 2(u_{h_0} + u_{h_1} \cdots + u_{h_{n-1}}) \geq 0, \; (h_0 < h_1 < \cdots < h_{n-1})
\end{cases} \tag{6}$$

where $h_0 = 0, 1, 2, \ldots, n,\; h_1 = 1, 2, \ldots, n, \ldots,\; h_{n-1} = n-1, n.$

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Let us find the vertices of the parallelohedron R. To this effect, let us examine a point (α_i) verifying the equations

$$n + 2u_1 = 0,\; 2(n-1) + 2(u_2 + u_2) = 0, \ldots, n \cdot 1 + 2(u_1 + u_2 + \ldots + u_n) = 0 \tag{7}$$

By virtue of (5), one obtains

$$\alpha_1 = \frac{1}{2}n,\; \alpha_2 = \frac{1}{2}(n-2),\; \alpha_k = \frac{1}{2}(n - 2k + 2), \ldots, \alpha_n = \frac{1}{2}(-n+2). \tag{8}$$

I argue that the point obtained (α_i) presents a vertex of the parallelohedron R. To demonstrate this, let us examine the form

$$f(x_1, x_2, \ldots, x_n) + 2\sum_{i=1}^{n} \alpha_i x_i$$

or, by (8), the form

$$f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{n} (n - 2i + 2)x_i.$$

For the point (α_i) determined by the equalities (8) to be a vertex of the parallelohedron R, it is necessary and sufficient that the inequality

$$f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{n} (n - 2i + 2)x_i \geq 0 \tag{9}$$

holds in the set E. By noticing that

$$f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{n}(n-2i+2)x_i =$$
$$\sum_{i=1}^{n}(x_i^2 + x_i) + \sum\sum_{i<j} \left[(x_i - x_j)^2 + x_i - x_j\right],$$

one obtains the inequalities (9) because the inequalities

$$x_i^2 + x_i \geq 0,\ (x_i - x_j)^2 + x_i - x_j \geq 0, \quad (i = 1, 2, \ldots, n; j = 2, 3, \ldots, n)$$

take place within the set E.

For the equality

$$f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{n}(n-2i+2)x_i = 0 \tag{10}$$

to hold, it is necessary and sufficient that one had the equality

$$x_i^2 + x_i = 0,\ (x_i - x_j)^2 + x_i - x_j = 0. (\mathrm{i} = 1, 2, \ldots, \mathrm{n}; \mathrm{i} < \mathrm{j}; \mathrm{j} = 2, 3, \ldots, \mathrm{n})$$

One declares that

$$x_i = -1,\ (i = 1, 2, \ldots, \lambda) \quad x_i = 0.\ (i = \lambda + 1, \ldots, n)$$

By attributing to the index λ the values $\lambda = 0, 1, 2 \ldots, n$ one obtains $n+1$ systems verifying the equality (10).

$$(0, 0, \ldots, 0), (-1, 0, \ldots, 0), (-1, -1, 0, \ldots, 0), \ldots, (-1, -1, \ldots, -1).$$

It is thus demonstrated that the point (α_i) determined by the equations (7) presents a simple vertex of the parallelohedron R.

Let us introduce in our studies symbol

$$(h_0, h_1, h_2, \ldots, h_n) \tag{11}$$

in which the indices $h_0, h_1, h_2, \ldots, h_n$ present a permutation of numbers $0, 1, 2, \ldots, n$ and let us agee to indicate by this symbol a point which verifies the equations

$$\begin{aligned} n + 2u_{h_0} = 0,\ 2(n-1) + 2\left(u_{h_0} + u_{h_1}\right) = 0, \ldots, \\ n + 2\left(u_{h_0} + u_{h_1} + \ldots, u_{h_{n-1}}\right) = 0. \end{aligned} \tag{12}$$

By virtue of the definition established of the symbol (11), the vertex (α_i) of the parallelohedron R determined by the equations (7) will be characterised by the symbol

$$(1, 2, \ldots, n, 0).$$

I argue that each symbol (11) characterises a vertex of parallelohedron R.

To demonstrate this, let us effect a transformation of the parallelohedron R with the help of a substitution

$$u_1 = u'_{h_0},\ u_2 = u'_{h_1}, \ldots, u_n = u'_{h_{n-1}},\ u_0 = u'_{h_n}, \tag{13}$$

where one has admitted

$$u'_0 = x'_1 + x'_2 + \ldots + x'_n,\ u'_1 = -x'_1, \ldots,\ u'_n = -x'_n.$$

The inequalities (6) which define the parallelohedron R will be permuted by the substitution considered, therefore the parallelohedron R will be transformed into itself.

To vertex (α_i) of the parallelohedron R determined by the equations (7) will be transformed, by virtue of (13), into a vertex of the parallelohedron R determined by the equations (12), therefore the vertex will be characterised by the symbol (11).

We have demonstrated the existence of $(n+1)!$ simple vertices of the parallelohedron R corresponding to the positive quadratic form φ. As the number of vertices of any one parallelohedron corresponding to a positive quadratic form does not exceed a limit $(n+1)!$, by virtue of the formula (3) of Number 101, one concludes that the parallelohedron R does not possess vertices other than those which are characterised by the symbol

$$(h_0, h_1, \ldots, h_n)$$

in which one permutes the indices $0, 1, 2, \ldots, n$ in every possible ways.

All the vertices of the parallelohedron R are simple, therefore the parallelohedron R is primitive. By noticing that the number of vertices of the parallelohedra R is expressed by the formula

$$S_0 = (n+1)! = (n+1)\Delta^{(m)}(m^n)_{m=1}, \tag{14}$$

one concludes, by virtue of that which has been said in Number 66, that the number S_ν of faces in ν dimensions of the parallelohedron R is expressed by the formula

$$S_\nu = (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \ldots, n)$$

Let us find the regulators and the characteristics of faces in $n-1$ dimensions of simplexes of the set (L) which defines the type of primitive parallelohedra to which belongs the parallelohedron R examined.

Any symbol $(h_0, h_1, \ldots, h_n)$ defines a simplex characterised by the linear functions

$$u_{h_0}, u_{h_0} + u_{h_1}, \ldots, u_{h_0} + u_{h_1} + \ldots + u_{h_n}.$$

By virtue of (5), one will have identically

$$u_{h_0} + u_{h_1} + \ldots + u_{h_n} = 0. \tag{15}$$

Notice that $n+1$ simplexes which one obtains by carrying out the circular permutations of indices $h_0, h_1, \ldots, h_n$

$$(h_0, h_1, \ldots, h_n), (h_1, h_2, \ldots, h_0), \ldots, (h_n, h_0, \ldots, h_{n-1})$$

are congruent. By choosing a representative among these simplexes, one will determine in this manner $n!$ incongruent simplexes of the set (L).

Let us examine two simplexes determined by two symbols

$$(h_0, h_1, h_2, \ldots, h_n) \quad \text{and} \quad (h_1, h_0, h_2, \ldots, h_n)$$

which differ only by a transposition of indices h_0 and h_1.

By virtue of the definition established, these simplexes are characterised by the functions

$$[u_{h_0}, u_{h_0} + u_{h_1}, u_{h_0} + u_{h_1} + u_{h_2}, \ldots, u_{h_0} + u_{h_1} + \ldots + u_{h_n}] \tag{16}$$

and

$$[u_{h_1}, u_{h_1} + u_{h_0}, u_{h_1} + u_{h_0} + u_{h_2}, \ldots, u_{h_1} + u_{h_0} + \ldots + u_{h_n}]. \tag{17}$$

These two simplexes differ only by the vertices which are characterised by the function u_{h_0} and u_{h_1}.

One concludes that these two simplexes are contiguous by a face in $n-1$ dimensions which is characterised by the functions

$$[u_{h_0} + u_{h_1}, u_{h_0} + u_{h_1} + u_{h_2}, \ldots, u_{h_0} + u_{h_1} + \ldots + u_{h_n}]. \tag{[1]8}$$

Let us determine the characteristic $\pm(p_i)$ of this face. By declaring, as that which we have done in Number 72,

$$u^0_{h_0} + u^0_{h_1} = \delta, u^0_{h_0} + u^0_{h_1} + u^0_{h_2} = \delta, \ldots, u^0_{h_0} + \ldots, + u^0_{h_1} + u^0_{h_n} = \delta,$$

one obtains, by (15), $\delta = 0$ and consequently

$$u^0_{h_0} = -u^0_{h_1}, u^0_{h_2} = 0, \ldots, u^0_{h_n} = 0. \tag{19}$$

By indicating with (p_i) the characteristic of the face (18) with regard to the simplex (16), one will have a supplementary condition

$$u^0_{h_0} > 0 \tag{20}$$

which, added to the equalities (19), well defines the characteristic (p_i).

Let us indicate, to make short,

$$h_0 = i \quad \text{and} \quad h_1 = j \tag{21}$$

and suppose that $i \neq 0$ and $j \neq 0$. By virtue of equalities (5), one will have

$$u^0_{h_0} = u^0_i = -p_i, u^0_{h_1} = u^0_j = -p_j.$$

By virtue of (19) and (20), one obtains

$$\begin{cases} p_k = 0, \\ p_i = -1, p_j = 1. \end{cases} \quad (k = 1, 2, \ldots, n; k \neq i; k \neq j) \tag{22}$$

One can therefore characterise the characteristic (p_i) by a corresponding function

$$\sum p_i x_i = -x_i + x_j.$$

Let us suppose that $j = 0$. One will have in this case the equalities

$$p_1 + p_2 + \ldots + p_n = p_i, p_k = 0 \quad (k = 1, 2, \ldots, n; k \neq i)$$

and consequently, by (20),

$$\sum p_i x_i = -x_i.$$

In the same way one will examine the case $i = 0$. One can bring together the three cases examined by indicating the characteristic of the face (18) by the function $-x_i + x_j$, provided that $x_0 = 0$.

Let us find the regulator ρ_{ij} of the face examined. To that effect, let us determine the number $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ after the conditions

$$u_{h_1} = \sum_{\lambda=0}^{n} \vartheta_\lambda (u_{h_0} + \ldots + u_{h_\lambda}) \text{ where } \sum \vartheta_\lambda = 1.$$

One obtains

$$\vartheta_0 = -1, \vartheta_1 = 1, \vartheta_2 = 0, \ldots, \vartheta_{n-1} = 0, \vartheta_n = 1.$$

By applying the formula (20) of Number 72, one finds

$$2\rho_{ij}\left[(u^0_{h_j}+\ldots+u^0_{h_\lambda})-u^0_{h_1}\right]=(u_{h_1})^2+(u_{h_0})^2-(u_{h_0}+u_{h_1})^2$$
$$-(u_{h_0}+u_{h_1}+\ldots+u_{h_n})^2 \quad \text{where} \quad \lambda>0.$$

By virtue of equalities (15) and (19), this formula comes down to the one here

$$2\rho_{ij}=(u_{h_1})^2+(u_{h_0})^2-(u_{h_0}+u_{h_1})^2,$$

and consequently

$$\rho_{ij}=-u_{h_0}u_{h_1}$$

or, by (21)

$$\rho_i=-u_iu_j. \tag{23}$$

Let us suppose that $j=0$; the formulae (5) give

$$\rho_{i0}=x_i(x_1+x_2+\ldots+x_n)=x_1x_i+x_2x_i+\ldots+x_nx_i.$$

By replacing in this formula x_ix_j by a_{ij}, one obtains the sought-for expression of the regulator ρ_{i0}

$$\rho_{i0}=\sum_{k=1}^{n}a_{ki} \quad (i=1,2,\ldots,n) \tag{24}$$

By supposing that $i\neq 0$ and $j\neq 0$, one will have

$$\rho_{ij}=-x_ix_j$$

and consequently

$$\rho_{ij}=-a_{ij} \quad (i=1,2,\ldots,\nu;i\neq j;j=1,2,\ldots,n) \tag{25}$$

Observe that the face (18) possesses the regulator ρ_{ij} and the characteristic $-x_i+x_j$ in addition to values of indices $h_2,\ldots,h_n$. One concludes that there exist $(n-1)!$ inconcruent faces of simplexes of the set (L) which possess the same regulator ρ_{ij} and the same characteristic

$$-x_0+x_j. \quad (i=0,1,2,\ldots,n;i\neq j;j=0,1,2,\ldots,n)$$

By applying the formula (I.) from Number 84, one obtains

$$\sum\sum a_{ij}x_ix_j=\sum_{i<j}\sum\rho_{ij}(x_i-x_j)^2,$$
$$(i=0,1,2,\ldots,n;i<j;j=1,2,\ldots,n)$$

where one has admitted $x_0=0$, or differently

$$\sum\sum a_{ij}x_ix_j=\sum_{i=1}^{n}\rho_{i0}x_i^2+\sum_{i<j}\sum\rho_{ij}(x_i-x_j)^2. \tag{26}$$
$$(i=1,2,\ldots,n;i<j;j=2,3,\ldots,n)$$

The domain Δ of quadratic forms corresponding to the type of primitive parallelohedra examined will be determined by the inequalities

$$\rho_{ij}\geq 0 \quad (i=0,1,2,\ldots,n;j=0,1,2,\ldots,n)$$

or differently, according to (24) and (25), by the inequalities

$$\sum_{k=1}^{n}a_{ki}\geq 0, \quad -a_{ij}\geq 0. \quad (i=1,2,\ldots,n;i<j;j=2,3,\ldots,n) \tag{27}$$

The number of these inequalities is equal to $\frac{n(n+1)}{2}$, thus the domain of quadratic forms defined by these inequalities is a simple domain.

By attributing to the parameters ρ_{ij} $(i=0,1,2,\ldots,n;j=0,1,2,\ldots,n;i<j)$ in the formula (26) the positive arbitrary values or zeros, one will determine all the quadratic forms belonging to the domain Δ.

One remarkable coincidence is signalling. The domain of quadratic forms (27) has been studied in my first mémoire cited † where it has been called principal domain. This domain corresponds to a principal perfect positive quadratic form

$$\varphi=x_1^2+x_2^2+\ldots+x_n^2+x_1x_2+\ldots+x_{n-1}x_n.$$

It is remarkable that the set of characteristics found

$$\pm x_i, \ \pm(x_i-x_j), \ (i=1,2,\ldots,n;i<j;j=2,3,\ldots,n)$$

coincides with the set of representations of the minimum of the principal perfect form φ.

105 *Domains of quadratic forms contiguous to the principal domain.*

† This Journal, V. 133

All the faces in $\frac{n(n+1)}{2} - 1$ dimensions of the principal domain Δ are equivalent. †

It follows that all the domains of forms belonging to the set (Δ) which are contiguous to the principal domain Δ by the faces in $\frac{n(n+1)}{2} - 1$ dimensions are equivalent.

In the case $n = 2$ and $n = 3$, the set (Δ) of domains of quadratic forms is made up of a single class of domains equivalent to the principal domain.

One concludes that in the space of 2 and of 3 dimensions there is only a single type of primitive parallelohedra, provided that one does not consider as different the equivalent types which correspond to the equivalent domains of quadratic forms.

Let us suppose that $n \geq 4$ and find the domain Δ' which is contiguous to the principal domain Δ by the face determined with the help of the equation

$$\rho = -a_{12} = 0.$$

By applying the algorithm explained in Number 96, let us determine the incongruent convex polyhedra which correspond to the independent regulator ρ.

We have seen in Number 104 that the regulator $\rho = \rho_{12}$ corresponds to the common faces of simplexes defined by the symbols

$$(1, 2, h_1, \ldots, h_n), (2, 1, h_2, \ldots, h_n)$$

where $h_2, h_3, \ldots, h_n$ present an arbitrary permutation of indices $0, 3, 4, \ldots, n$.

The two corresponding simplexes are characterised by the functions

$$[u_1, u_1 + u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1 + u_2 + \ldots + u_{h_n}]$$

and

$$[u_2, u_2 + u_1, u_2 + u_1 + u_{h_2}, \ldots, u_2 + u_1 + \ldots + u_{h_n}].$$

By declaring

$$u_2 = \vartheta_0 u_1 + \vartheta_1(u_1 + u_2) + \vartheta_2(u_1 + u_2 + u_{h_2}) + \ldots + \vartheta_n(u_1 + u_2 + \ldots + u_{h_n})$$

where $\sum_{k=0}^{n} = 1$, one obtains

$$\vartheta_0 = -1, \vartheta_1' = 1, \vartheta_2 = 0, \ldots, \vartheta_{n-1} = 0, \vartheta_n = 1. \tag{3}$$

As among the numbers obtained is found only one negative number ϑ_0, one concludes that the two simplexes (1) and (2) make up a polyhedron K corresponding to the independent regulator ρ.

Let us indicate by (L') the set of simplexes which characterise the domain Δ' of quadratic forms. By virtue of that which has been said in Number 91, the polyhedron K in the set (L') will be made up from simplexes which one obtains by replacing the vertices of the simplex (1) which correspond to the positive values of numbers (3) by the vertex characterised by the function u_2. As in the series (3) only two positive numbers ϑ_1 and ϑ_n are found, one obtains two simplexes characterised by the functions

$$[u_1, u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1 + u_2 + \ldots + u_{h_n}] \tag{4}$$

and

$$\left[u_1, u_1 + u_2, \ldots, u_1 + u_2 + \ldots + u_{h_{n-1}}, u_2\right]. \tag{5}$$

These two simplexes make up the polyhedron K and replace the two simplexes (1) and (2) in the set (L').

By effecting all the permutation of indices $h_2, \ldots, h_n$, one obtains $(n-1)!$ incongruent convex polyhedra which correspond to the independent regulator ρ.

By replacing in the set (L) the simplexes congruent to the simplexes (1) and (2) by the simplexes which are congruent to the simplexes (4) and (5), one will reconstruct the set (L) of simplexes into a set (L') which characterises the domain Δ'.

Notice that the number of incongruent simplexes of the set (L') is equal to $n!$ also. It follows that the number of faces in ν dimensions of primitive parallelohedra belonging to the type characterised by the domain of form Δ' is expressed by the formula

$$S_\nu = (n + 1 - \nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \ldots, n)$$

106 Let us find the regulators and the characteristic of faces in $n-1$ dimensions of simplexes belonging to the set (L').

Let us examine in the first place the simplexes contiguous to a face in $n-1$ dimensions which belong to the set (L) and to the set (L').

The condition necessary and sufficient for which the two simplexes characterised by the symbols

$$(h_0, h_1, h_2, \ldots, h_n) \text{ and } (h_1, h_0, h_2, \ldots, h_n), \tag{6}$$

which are contiguous by a face in $n-1$ dimensions, also belong to the set (L'), consists in so long as within the two series

$$h_0, h_1, h_2, \ldots, h_n, h_0 \quad \text{and} \quad h_1, h_0, h_2, \ldots, h_n, h_1$$

the indices 1 and 2 are not adjacent. By declaring

$$h_0 = i \quad \text{and} \quad h_1 = j,$$

one obtains $(n-1)! - 2(n-2)!$ pairs of symbols (6) which satisfy the condition assumed.

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† See my mémoire cited

By indicating with ρ_{ij} the regulator and with $\pm(x_i - x_j)$ the characteristic of the face common to the simplexes (6) determined in the set (L), one will have for the set (L') the same regulator

$$\rho'_{ij} = \rho_{ij}, \quad (i = 0, 1, 2, \ldots, n; i < j; j = 1, 2, \ldots, n)$$

and the same characteristic $\pm(x_i - x_j)$, the regulator ρ_{12} being excluded. The regulator obtained ρ'_{ij} and the characteristic $\pm(x_i - x_j)$ belong to $(n-1)! - 2(n-2)!$ incongruent faces in $n-1$ dimensions of the set (L') of simplexes.

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This declared, let us examine the regulators and the characteristics of faces of simplexes (4) and (5) which make up the polyhedron K in the set (L').

The first group of faces of the simplex (4) is composed of two faces which are opposite to the vertices u_1 and u_2. The first face is characterised by the functions

$$u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1, u_2, \ldots, u_{h_n}. \tag{7}$$

This face belongs in the set (L) to the simplexes characterised by the symbols

$$(2, 1, h_2, h_3, \ldots, h_n) \quad \text{and} \quad (2, h_2, 1, h_3, \ldots, h_n).$$

The second simplex also belongs to the set (L'). It follows that the simplex (4) is contiguous to the simplex $(2, h_2, 1, h_3, \ldots, h_n)$ by the face (7).

Let us declare $h_2 = i$ where $i = 0, 3, \ldots, n$ and indicate by ρ'_{i1} the regulator and by $\pm(x_1 - x_i)$ the characteristic of the face (7) in the set (L'). By applying the formula (8) of Number 93, one obtains

$$\rho'_{i1} = \rho_{i1} + \rho. \ \ (i = 0, 3, \ldots, n)$$

In the same manner, one will examine the regulators of the face of the simplex (4) which is opposite to the vertex u_2. By putting $h_2 = i$, one will have

$$\rho'_{i2} = \rho_{i2} + \rho. \ \ (i = 0, 3, \ldots, n) \tag{9}$$

Examine the first group of faces of the simplex (5). This group is made up of two faces which are opposite to the vertices u_1 and u_2. The first face is characterised by the functions.

$$u_1 + u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1 + u_2 + \ldots + u_{h_{n-1}}, u_2.$$

This face belongs in the set (L) to the simplex $(2, 1, h_2, \ldots, h_n)$ and to the simplex congruent to the simplex $(h_n, 1, h_2, \ldots, h_{n-1}, 2)$ and which is characterised by the functions

$$u_2, u_2 + u_1, u_2 + u_1 + u_{h_2}, \ldots, u_2 + u_1 + \ldots + u_{h_{n-1}}, u_2 - u_{h_n}.$$

This simplex also belongs to the set (L'). By putting $h_n = i$ and by applying the formula (8) of Number (93), one finds

$$\rho'_{i2} = \rho_{i2} + \rho$$

The characteristic of this face will be $\pm(x_2 - x_i)$ $(i = 0, 3, \ldots, n)$.

In the same way one will examine the face of the simplex (5) which is opposite to the vertex u_2.

One will obtain by letting $h_n = i$

$$\rho'_{i1} = \rho_{i1} + \rho$$

and the characteristic will be $\pm(x_1 - x_i)$, $(i = 0, 3, \ldots, n)$.

Let us notice that the number of incongruent faces, belonging to the first group of simplexes of the set (L'), which possess the regulator determined by the formula (8) or by the formula (9), is equal to $2(n-2)!$.

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The second group of faces in $n-1$ dimensions of the simplex (4) is composed of $n-2$ faces which are opposite to the vertices

$$u_1 + u_2 + u_{h_2}, u_1 + u_2 + u_{h_2} + u_{h_3}, \ldots, u_1 + u_2 + \ldots, u_{h_{n-1}}.$$

Let us examine a face which is opposite to the vertex $u_1 + u_2 + \ldots + u_{h_k}$ $(k = 2, 3, \ldots, n-1)$.

A transposition of indices h_k and h_{k+1} in the symbol $(*)$ leads to the symbol

$$[u_1, u_2, \ldots, u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_{k+1}}, u_1 + u_2 + \ldots + u_{h_{k+1}} + u_{h_k},$$
$$\ldots, u_1 + u_2 + \ldots + u_{h_n}]$$

which defines a simplex belonging to the set (L') and which is contiguous to the simplex (4) by the face in $n-1$ dimensions examined.

Let us write $h_k = i, h_{k+1} = j$ and indicate by ρ'_{ij} the corresponding regulator. The corresponding characteristic will be determined by the equations

$$u_1^0 = 0, u_2^0 = 0, \ldots, u_1^0 + u_2^0 + \ldots + u_{h_{k-1}}^0 = 0,$$
$$u_1^0 + u_2^0 + \ldots + u_{h_{k+1}}^0 = 0, u_1^0 + u_2^0 + \ldots + u_{h_n}^0 = 0.$$

One obtains

$$u_i^0 = -u_j^0 \ \text{and} \ u_r^0 = 0. \quad (r = 0, 1, 2, \ldots, n; r \neq i; r \neq j)$$

It follows that the characteristic will be represented by the function $\pm(x_i - x_j)$.

By declaring

$$\begin{aligned} u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_{k+1}} &= \vartheta_0 u_1 + \vartheta_1 u_2 + \vartheta_2 (u_1 + u_2 + u_{h_2}) + \ldots \\ &+ \vartheta_k (u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_k}) + \ldots + \vartheta_n (u_1 + u_2 + \ldots + u_{h_n}) \end{aligned} \tag{10}$$

where $\sum_{r=0}^{n} \vartheta_r = 1$, one obtains

$$\begin{aligned} \vartheta_0 = 0, \vartheta_1 = 0, \ldots, \vartheta_{k-2} = 0, \vartheta_{k-1} = 1, \vartheta_k = -1, \vartheta_{k+1} = 1, \vartheta_{k+2} = 0, \\ \ldots, \vartheta_n = 0, \end{aligned}$$

provided that $k > 2$.

The regulator ρ'_{ij} will be determined by the formula

$$\begin{aligned} 2\rho'_{ij} =& (u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_{k+1}})^2 - (u_1 + u_2 + \ldots + u_{h_{k-1}})^2 + \\ & (u_1 + u_2 + \ldots + u_{h_k})^2 - (u_1 + u_2 + \ldots + u_{h_{k+1}})^2. \end{aligned}$$

After the reductions, one finds

$$\rho'_{ij} = -u_{h_k} u_{h_{k+1}}$$

or differently

$$\rho'_{ij} = -u_i u_j,$$

thus, by virtue of the formula (23) of Number 104, one will have

$$\rho'_{ij} = \rho_{ij}. \quad (i = 0, 3, \ldots, n; j = 0, 3, \ldots, n; k = 3, 4, \ldots, n-1) \tag{11}$$

Let us examine the case $k = 2$. The equality (10) gives in this case

$$\vartheta_0 = 1, \vartheta_1 = 1, \vartheta_2 = -1, \vartheta_3 = 1, \vartheta_4 = 0, \ldots, \vartheta_{n-1} = 0, \vartheta_n = -1,$$

and consequently

$$\begin{aligned} 2\rho'_{ij} =& (u_1 + u_2 + u_{h_3})^2 - u_1^2 - u_2^2 + (u_1 + u_2 + u_{h_2})^2 \\ & - (u_1 + u_2 + u_{h_2} + u_{h_3})^2 + (u_1 + u_2 + \ldots + u_{h_n})^2. \end{aligned}$$

As $u_{h_2} = i$, $u_{h_3} = j$ and $u_1 + u_2 + \ldots + u_{h_n} = 0$, one obtains

$$\rho'_{ij} = u_1 u_2 - u_i u_j,$$

thus one will have in the case examined

$$\rho'_{ij} = \rho_{ij} - \rho. \quad (i = 0, 3, \ldots, n; j = 0, 3, \ldots, n) \tag{12}$$

In the same way, one will examine the faces of the simplex (5) belonging to the second group and one will obtain the same formulae (11) and (12).

Let us notice that the number of incongruent faces of simplexes of the set (L') which belong to the second group and the regulator of which is determined by the formula (11) is equal to $(n-3)!2(n-3)$. The number of regulators which are determined by the formula (12) is equal to $2(n-3)!$.

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The third group of faces of simplexes (4) and (5) is composed of a single face

$$\left[u_1, u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1 + u_2 + \ldots + u_{h_{n-1}}\right]$$

which is common to these two simplexes.

The characteristic of this face is determined by the equations

$$u_1^0 = \delta, u_2^0 = \delta, u_1^0 + u_2^0 + u_{h_2}^0 = \delta, \ldots, u_1^0 + u_2^0 + \ldots + u_{h_{n-1}}^0 = \delta.$$

It results in that

$$u_1^0 = u_2^0 = \delta, u_2^0 = -\delta, u_{h_3}^0 = 0, \ldots, u_{h_{n-1}}^0 = 0, u_{h_n}^0 = \delta.$$

One concludes that $\delta = \pm 1$. By admitting

$$h_2 = i \quad \text{and} \quad h_n = j,$$

one obtains the characteristic $\pm(x_1 + x_2 - x_i - x_j)$.

Let us indicate the corresponding regulator by ρ'_{ij}. with the help of equalities

$$\begin{aligned} u + u_2 =& \vartheta_0 u_1 + \vartheta_1 u_2 + \vartheta_2 (u_1 + u_2 + u_{h_2}) + \ldots \\ & + \vartheta_n (u_1 + u_2 + \ldots + u_{h_n}) \text{ where } \sum_{k=1}^{n} \vartheta_k = 1, \end{aligned}$$

one obtains

$$\vartheta_0 = 1, \vartheta_1 = 1, \vartheta_2 = 0, \ldots, \vartheta_{n-1} = 0, \vartheta_n = -1.$$

The regulator ρ'_{ij} will be determined by the formula

$$2\rho'_{ij} = (u_1 + u_2)^2 - u_1^2 - u_2^2 - (u_1 + u_2 + \ldots + u_{h_n}),$$

and it becomes

$$\rho'_{ij} = u_1 u_2,$$

thus

$$\rho'_{ij} = -\rho. \quad (i = 0, 3, \ldots, n; j = 0, 3, \ldots, n)$$

The number of incongruent faces belonging to the third group having the characteristic $\pm(x_1+x_2-x_i-x_j)$ is equal to $2(n-3)!$.

110 With the help of deduced formulae, one can determine all the independent regulators. Let us admit

$$\rho'_{12} = -\rho,\ \rho'_{i1} = \rho i1 + \rho,\ \rho'_{i2} = \rho i2 + \rho,\ \rho'_{ij} = \rho_{ij}. \tag{13}$$
$$(i = 0, 3, \ldots, n; i < j; j = 3, \ldots, n)$$

One obtains in this manner $\frac{n(n+1)}{2}$ independent regulators ρ'_{ij} $(i = 0, 1, 2, \ldots, n; i < j; j = 1, 2, \ldots, n)$.

The results of our studies can be gathered in the following table:

1. Regulator ρ'_{12}, characteristic $\pm(x_1+x_2-x_i-x_j)$, their number $2(n-3)!$
2. Regulator ρ'_{i1}, characteristic $\pm(x_1-x_i)$, their number $2(n-2)!$
3. Regulator ρ'_{i2}, characteristic $\pm(x_2-x_i)$, their number $2(n-2)!$
4. Regulator $\rho'_{i1}+\rho'_{12}$, characteristic $\pm(x_1-x_i)$, their number $(n-1)!-2(n-2)!$
5. Regulator $\rho'_{i2}+\rho'_{12}$, characteristic $\pm(x_2-x_i)$, their number $(n-1)!-2(n-2)!$
6. Regulator ρ'_{ij}, characteristic $\pm(x_i-x_j)$, their number $(n-1)!-2(n-3)!$
7. Regulator $\rho'_{ij}+\rho'_{12}$, characteristic $\pm(x_i-x_j)$, their number $2(n-3)!$

The indices i and j are the values $0, 3, \ldots, n$ and one has admitted $x_0 = 0$.

The domain Δ' of quadratic forms corresponding to the set (L') of simplexes is determined by $\frac{n(n+1)}{2}$ independent inequalities

$$\rho'_{ij} \geq 0. \quad (i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, n; i \neq j)$$

As a result, the domain Δ' is simple.

By applying the formula (I) of Number 84, one will determine, by (13), any quadratic form $\sum\sum a_{ij}x_ix_j$ by the following formula

$$\sum\sum a_{ij}x_ix_j = \sum\sum_{i<j}\rho'_{ij}(x_i-x_j)^2 + \rho'_{12}\omega, \tag{14}$$
$$(i = 0, 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

where one has admitted $x_0 = 0$ and

$$\omega = (n-2)x_1^2 + (n-2)x_2^2 + 2x_3^2 + \ldots + 2x_n^2 + 2x_1x_2 + 2x_1x_3 - \ldots$$
$$- 2x_1x_n - 2x_2x_3 - \ldots - 2x_2x_n.$$

By attributing to the independent parameters ρ'_{ij} $(i = 0, 1, 2, \ldots, n;\ j = 0, 1, 2, \ldots, n)$ all the positive values or null, one will determine by the formula (14) all the quadratic forms belonging to the domain Δ'.

One coincidence is to be pointed out: the domain Δ' presents a part of the domain R, corresponding to the perfect form φ, which has been determined in my mémoire cited. The set composed of linear forms

$$\pm(x_i - x_j) \quad (i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, n)$$

(the form $\pm(x_1 - x_2)$ being excluded) and of forms

$$(x_1 + x_2 - x_i - x_j) \quad (i = 0, 3, \ldots, n; j = 0, 3, \ldots, n)$$

where one has put $x_0 = 0$, coincides with the linear forms which define all the representations of the minimum of the perfect form φ_1.

Parallelohedra in two dimensions

111 The set (Δ) of domains of binary quadratic forms is composed of a single class of domains which are equivalent to the principal domain Δ determined by the inequalities

$$a + b \geq 0, \quad -b \geq 0, \quad c + b \geq 0.$$

Here are the conditions of reduction of binary positive quadratic forms $ax^2 + 2bxy + cy^2$ due to Selling. ‡ Any quadratic form belonging to the principal domain Δ can be determined by the equalities

$$ax^2 + 2bxy + cy^2 = \lambda x^2 + \mu y^2 + \nu(x-y)^2 \text{ where } \lambda > 0,\ \mu \geq 0,\ \nu \geq 0.$$

The parameters λ, μ and ν present the regulators of the hexagon of Lejeune Dirichlet ‡ defined by the inequalities

$$-\frac{1}{2}(\lambda+\nu) \leq x \leq \frac{1}{2}(\lambda+\nu),$$
$$-\frac{1}{2}(\mu+\nu) \leq y \leq \frac{1}{2}(\mu+\nu),$$
$$-\frac{1}{2}(\lambda+\mu) \leq x+y \leq \frac{1}{2}(\lambda+\mu),$$

‡ See the Introduction

By attributing to the arbitrary parameters λ, μ, ν the positive values, one will determine by these inequalities a primitive parallelohedron in two dimensions, that is to say a hexagon of Lejeune Dirichlet.

By nullifying one of the parameters λ, μ, ν, for example ν, one will obtain four independent inequalities.

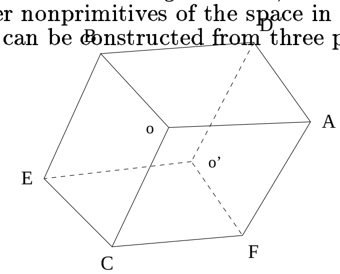
$$-\frac{1}{2}\lambda \leq x \leq \frac{1}{2}\lambda,$$
$$-\frac{1}{2}\mu \leq y \leq \frac{1}{2}\mu,$$

which define a nonprimitive parallelohedron in two dimensions, it is a parallelogram.

It is easy to demonstrate that other nonprimitives of the space in two dimensions do not exist.

Each hexagon of Lejeune Dirichlet can be constructed from three parallelograms as is indicated in Fig. 1.

Fig. 1



One of the three parallelograms $OADB$, $OBEC$ and $OCFA$ which form the hexagon $ADBECF$ can be arbitrarily chosen. By choosing, for example, an arbitrary parallelogram $OADB$, one will determine the two remaining parellograms $OBEC$ and $OCFA$ by taking an arbitrary vector OC, provided that by extending this vector in the inverse direction one passes through the chosen parallelogram $OADB$.

Observe that in general the point O does not present the centre of the hexagon $ADBECF$.

One can make up the same hexagon of three parallelograms $O'DBE$, $O'ECF$ and $O'FAD$. One concludes that the hexagon of Lejeune Dirichlet does not present anything other than projection of a parallelepiped on the plane.

Parallelohedra in three dimensions

112 The set (Δ) of domains of ternary quadratic forms is composed of a single class of domains equivalent to the principal domain Δ determined by the inequalities

$$a + b' + b'' \geq 0, a' + b'' + b \geq 0, a'' + b + b' \geq 0, -b \geq 0, -b' \geq 0, -b'' \geq 0.$$

Here are the conditions of reduction of ternary positive quadratic forms $ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy$ due to Selling.

Any ternary quadratic form belonging to the principal domain Δ can be determined by the equalities

$$\begin{aligned} ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy = \\ \lambda x^2 + \lambda' y^2 + \lambda'' z^2 + \mu(y-z)^2 + \mu'(z-x)^2 + \mu''(x-y)^2. \end{aligned}$$

All the primitive parallelohedra in three dimensions can be transformed with the help of linear substitutions into primitive parallelohedra determined by 14 independent inequalities

$$\left\{ \begin{array}{lrcll} (1') & -\frac{1}{2}(\lambda + \mu' + \mu'') & \leq x \leq & \frac{1}{2}(\lambda + \mu' + \mu''), & (1'), \\ (2') & -\frac{1}{2}(\lambda' + \mu'' + \mu) & \leq y \leq & \frac{1}{2}(\lambda' + \mu'' + \mu), & (2'), \\ (3') & -\frac{1}{2}(\lambda'' + \mu + \mu') & \leq z \leq & \frac{1}{2}(\lambda'' + \mu + \mu'), & (3'), \\ (4') & -\frac{1}{2}(\lambda' + \lambda'' + \mu' + \mu'') & \leq y + z \leq & \frac{1}{2}(\lambda' + \lambda'' + \mu' + \mu''), & (4'), \\ (5') & -\frac{1}{2}(\lambda_{,,} + \lambda + \mu_{,,} + \mu) & \leq z + x \leq & \frac{1}{2}(\lambda_{,,} + \lambda + \mu_{,,} + \mu), & (5'), \\ (6') & -\frac{1}{2}(\lambda + \lambda_{,} + \mu + \mu') & \leq x + y \leq & \frac{1}{2}(\lambda + \lambda_{,} + \mu + \mu'), & (6'), \\ (7') & -\frac{1}{2}(\lambda + \lambda' + \lambda'') & \leq x + y + z \leq & \frac{1}{2}(\lambda + \lambda' + \lambda''), & (7') \end{array} \right. \quad (2)$$

The parameter $\lambda, \lambda_{,}, \lambda_{,,}, \mu, \mu_{,}, \mu_{,,}$ present the independent regulators of the primitive parallelohedron defined by these inequalities and corresponding to a ternary positive quadratic form (1).

By virtue of the formula (14) of Number 103, any primitive parallelohedron of the space in three dimensions possesses 24 vertices which can be characterised by three numbers corresponding to the different faces in two dimensions of the parallelohedron defined by the inequalities (2).

One will divide these (24) vertices into three groups I, II and III:

I	1 6 7	1′ 2 4	2′ 6′ 3	4′ 3′ 7′
	1′ 6′ 7′	1 2′ 4′	2 6 3′	4 3 7
II	1 7 5	1′ 4 3	4′ 7′ 2′	3′ 2 5′
	1′ 7′ 5′	1 4′ 3′	4 7 2	3 2′ 5
III	1 3′ 6	1′ 5′ 2	5 3 7	2′ 7′ 6′
	1′ 3 6′	1 5 2′	5′ 3′ 7′	2 7 6

Each line of this table is composed of four congruent vertices. In each group the second line is formed from vertices opposite to those which are found in the first line.

Let us examine the regulators and the characteristics of edges of the primitive parallelohedron in three dimensions. It suffices to examine the regulators and the characteristics of faces in two dimensions belonging to three simplexes

$$(0, 1, 6, 7), \quad (0, 1, 7, 5), \quad (0, 1, 3', 6).$$

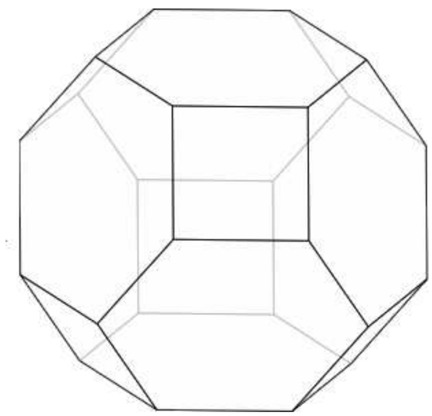
The results of these studies can be brought together in the following table:

I	0 1 6 7	−1 0 0 2 1 1	λ	1 −1 0 0 1 0	μ''	0 1 −1 1 0 1	μ	0 0 1 0 0 −1	λ''
II	0 1 7 5	−1 0 0 2 1 1	λ	1 0 −1 0 0 1	μ'	0 1 0 0 −1 0	λ'	0 −1 1 1 1 0	μ
III	0 1 3′ 6	−1 0 1 1 0 −1	μ'	1 −1 0 0 1 9	μ''	0 0 −1 1 1 1	$\lambda_{\prime\prime}$	0 1 0 0 −1 −1	$\lambda_{\prime}$

The first line contains the characteristics and the regulators of different faces of the corresponding simplexes. The second line is composed of vertices which define the simplexes contiguous to the corresponding simplex by the faces the characteristics of which are found above, in the first line.

The faces of primitive parallelohedron R in three dimensions (Fig. 2) are divided into 8 hexagons of Lejeune Dirichlet and into 6 parallelograms.

Fig. 2



The hexagonal faces of the primitive parallelohedron R are characterised by the numbers

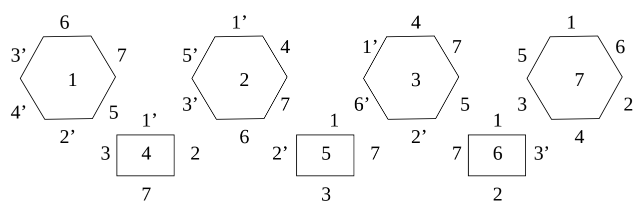
$$1, 2, 3, 7, 1', 2', 3', 7'.$$

The parallelogrammatic faces are characterised by the numbers

$$4, 5, 6, 4', 5', 6'.$$

The faces, the edges and the vertices of the primitive parallelohedron are systematically characterised in Fig. 3.

Fig. 3



One has indicated in this figure the numbers of faces which are contiguous to one of 7 incongruent faces. Each edge is characterised by two adjacent numbers and each vertex by three numbers.

113 By nullifying one or more parameters $\lambda, \lambda', \lambda'', \mu, \mu', \mu''$ in the inequalities (2), one will determine the nonprimitive parallelohedra in three dimensions. It is easy to see that the nonprimitive parallelohedra obtained are divided into four different spaces. (Fig. 4–7)

Fig. 4

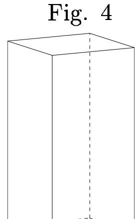


Fig. 5

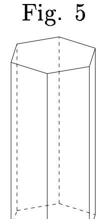


Fig. 6

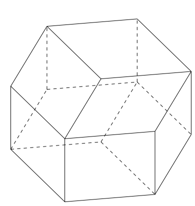
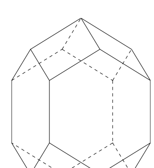


Fig. 7



Nonprimitive parallelohedra of the 1^{st} *space.* By making $\mu = 0, \mu' = 0, \mu'' = 0$ in the inequalities (2), one will obtain 6 independent inequalities

$$\begin{aligned} -\tfrac{1}{2}\lambda \quad &\le x \le \quad \tfrac{1}{2}\lambda, \\ -\tfrac{1}{2}\lambda' \quad &\le y \le \quad \tfrac{1}{2}\lambda', \\ -\tfrac{1}{2}\lambda'' \quad &\le z \le \quad \tfrac{1}{2}\lambda'', \end{aligned}$$

which define a parallelepiped (Fig. 4).

Nonprimirive parallelohedra of the 2^{nd} *space.* By making $\mu' = 0, \mu'' = 0$ in the inequalities (2), one will obtain 8 independent inequalities

$$\begin{array}{rcl} -\frac{1}{2}\lambda & \leq x \leq & \frac{1}{2}\lambda, \\ -\frac{1}{2}(\lambda' + \mu) & \leq y \leq & \frac{1}{2}(\lambda' + \mu), \\ -\frac{1}{2}(\lambda'' + \mu) & \leq z \leq & \frac{1}{2}(\lambda'' + \mu), \\ -\frac{1}{2}(\lambda' + \lambda'') & \leq y + z \leq & \frac{1}{2}(\lambda' + \lambda''), \end{array}$$

which define a prism with hexagonal base (Fig. 5).

Nonprimirive parallelohedra of the 3^{rd} *space.* By making $\lambda'' = 0, \mu'' = 0$ in the inequalities (2), one will obtain 12 independent inequalities

$$\begin{array}{rcl} -\frac{1}{2}(\lambda + \mu') & \leq x \leq & \frac{1}{2}(\lambda + \mu'), \\ -\frac{1}{2}(\lambda' + \mu) & \leq y \leq & \frac{1}{2}(\lambda' + \mu), \\ -\frac{1}{2}(\mu + \mu') & \leq z \leq & \frac{1}{2}(\mu + \mu'), \\ -\frac{1}{2}(\lambda' + \mu') & \leq y + z \leq & \frac{1}{2}(\lambda' + \mu'), \\ -\frac{1}{2}(\lambda + \mu) & \leq z + x \leq & \frac{1}{2}(\lambda + \mu), \\ -\frac{1}{2}(\lambda + \lambda') & \leq x + y + z \leq & \frac{1}{2}(\lambda + \lambda'), \end{array}$$

which define a parallelogrammatic dodecahedron (Fig. 6).

Nonprimirive parallelohedra of the 4^{th} *space.* By making $\mu'' = 0$ in the inequalities (2), one will obtain 12 independent inequalities

$$\begin{array}{rcl} -\frac{1}{2}(\lambda + \mu') & \leq x \leq & \frac{1}{2}(\lambda + \mu'), \\ -\frac{1}{2}(\lambda' + \mu) & \leq y \leq & \frac{1}{2}(\lambda' + \mu), \\ -\frac{1}{2}(\lambda'' + \mu + \mu') & \leq z \leq & \frac{1}{2}(\lambda'' + \mu + \mu'), \\ -\frac{1}{2}(\lambda' + \lambda'' + \mu') & \leq y + z \leq & \frac{1}{2}(\lambda' + \lambda'' + \mu'), \\ -\frac{1}{2}(\lambda'' + \lambda + \mu) & \leq z + x \leq & \frac{1}{2}(\lambda'' + \lambda + \mu), \\ -\frac{1}{2}(\lambda + \lambda' + \lambda'') & \leq x + y + z \leq & \frac{1}{2}(\lambda + \lambda' + \lambda''), \end{array}$$

which define a dodecahedron in 4 hexagonal faces and in 8 parallelogrammatic face (Fig. 7). Mr. Fedorow has demonstrated that other parallelohedra in three dimensions do not exist. ¶

Parallelohedra in four dimensions

114

The first type of primitive parallelohedra in four dimensions is characterised by the principal domain Δ of quaternary quadratic forms which is determined by the independent inequalities

$$\begin{array}{c} \lambda_1 = a_{11} + a_{12} + a_{13} + a_{14} \geq 0, \quad \lambda_2 = a_{21} + a_{22} + a_{23} + a_{24} \geq 0, \\ \lambda_3 = a_{31} + a_{32} + a_{33} + a_{34} \geq 0, \quad \lambda_4 = a_{41} + a_{42} + a_{43} + a_{44} \geq 0, \\ \mu_1 = -a_{12} \geq 0, \quad \mu_2 = -a_{13} \geq 0, \quad \mu_3 = -a_{14} \geq 0, \\ \mu_4 = -a_{23} \geq 0, \quad \mu_5 = -a_{24} \geq 0, \quad \mu_6 = -a_{34} \geq 0. \end{array}$$

Any quaternary quadratic form

$$f(x_1, x_2, x_3, x_4)$$

belonging to the domain Δ can be determined by the equalities

$$\begin{aligned} f(x_1,x_2,x_3,x_4) = {} & \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \mu_1 (x_1 - x_2)^2 + \\ & \mu_2 (x_1 - x_3)^2 + \mu_3 (x_1 - x_4)^2 + \mu_4 (x_2 - x_3)^2 + \\ & \mu_5 (x_2 - x_4)^2 + \mu_6 (x_3 - x_4)^2. \end{aligned} \tag{1}$$

The corresponding parallelohedron is determined by 30 inequalities which one will write down in the form

$$\pm(l_1 x_1 + l_2 x_2 + l_3 x_3 + l_4 x_4) \leq \frac{1}{2} f(l_1, l_2, l_3, l_4). \tag{2}$$

The systems $\pm(l_1, l_2, l_3, l_4)$ and the corresponding values

$$f(l_1, l_2, l_3, l_4)$$

of the quadratic form (1) are given in the following table:

Ist *type of parallelohedra*

¶ See the Introduction

N	l_1	l_2	l_3	l_4	$f(l_1, l_2, l_3, l_4)$	$-l_1$	$-l_2$	$-l_3$	$-l_4$	$2'$
1	1	0	0	0	$\lambda_1 + \mu_1 + \mu_2 + \mu_3$	-1	0	0	0	$1'$
2	0	2	0	0	$\lambda_2 + \mu_1 + \mu_4 + \mu_5$	0	-1	0	0	$2'$
3	0	0	1	0	$\lambda_3 + \mu_2 + \mu_4 + \mu_6$	0	0	-1	0	$3'$
4	0	0	0	1	$\lambda_4 + \mu_3 + \mu_5 + \mu_6$	0	0	0	-1	$4'$
5	1	1	0	0	$\lambda_1 + \lambda_2 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	-1	-1	0	0	$5'$
6	1	0	1	0	$\lambda_1 + \lambda_3 + \mu_1 + \mu_3 + \mu_4 + \mu_6$	-1	0	-1	0	$6'$
7	1	0	0	1	$\lambda_1 + \lambda_4 + \mu_1 + \mu_2 + \mu_5 + \mu_6$	-1	0	0	-1	$7'$
8	0	1	1	0	$\lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_5 + \mu_6$	0	-1	-1	0	$8'$
9	0	1	0	1	$\lambda_2 + \lambda_4 + \mu_1 + \mu_3 + \mu_4 + \mu_6$	0	-1	0	-1	$9'$
10	0	0	1	1	$\lambda_3 + \lambda_4 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	0	0	-1	-1	$10'$
11	1	1	1	0	$\lambda_1 + \lambda_2 + \lambda_3 + \mu_3 + \mu_5 + \mu_6$	-1	-1	-1	0	$11'$
12	1	1	0	1	$\lambda_1 + \lambda_2 + \lambda_4 + \mu_2 + \mu_4 + \mu_6$	-1	-1	0	-1	$12'$
13	1	0	1	1	$\lambda_1 + \lambda_3 + \lambda_4 + \mu_1 + \mu_4 + \mu_5$	-1	0	-1	-1	$13'$
14	0	1	1	1	$\lambda_1 + \lambda_3 + \lambda_4 + \mu_1 + \mu_2 + \mu_3$	0	-1	-1	-1	$14'$
15	1	1	1	1	$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$	-1	-1	-1	-1	$15'$

By attributing to the parameters

$$\lambda_1,\ \lambda_2, \ldots,\ \mu_6$$

the arbitrary positive values, one will determine with the help of inequalities (2) all the primitive parallelohedra of the first type.

The primitive parallelohedra of the first type possess 120 vertices which can be divided into 12 groups composed of congruent vertices and of opposite vertices.

All these vertices are put together in the following table:

Vertices of primitive parallelohedron of Ist *type*

I	1 5 11 15	1' 2 8 14	2' 5' 3 10	8' 3' 11' 4	14' 10' 4' 15'
	1' 5' 11' 15'	1 2' 8' 4'	2 5 3' 10'	8 3 11 4'	14 10 4 15
II	1 5 3' 10'	1' 2 6' 13'	2' 5' 11' 15'	6 11 3 4'	13 15 4 10
	1' 5' 3 10	1 2' 6 13	2 5 11 15	6' 11' 3' 4	13' 15' 4' 10'
III	1 6 11 4'	1' 3 8 7'	3' 6' 2 13'	8' 2' 11' 15'	7 13 15 4
	1' 6' 11' 4	1 3' 8' 7	3 6 2' 13	8 2 11 15	7' 13' 15' 4'
IV	1 7 13 15	1' 4 10 14	4' 7' 3 8	10' 3' 13' 2	14' 8' 2' 15'
	1' 7' 13' 15'	1 4' 10' 14'	4 7 3' 8'	10 3 13 2'	14 8 2 15
V	1 5 12 15	1' 2 9 14	2' 5' 4 10	9' 4' 12' 3	14' 10' 3' 15'
	1' 5' 12' 15'	1 2' 9' 14'	2 5 4' 10'	9 4 12 3'	14 10 3 15
VI	1 5 4' 10'	1' 2 7' 13'	2' 5' 12' 15'	7 12 4 3'	13 15 3 10
	1' 5' 4 10	1 2' 7 13	2 5 12 15	7' 12' 4' 3	13' 15' 3' 10'
VII	1 7 12 3'	1' 4 9 6'	4' 7' 2 13'	9' 2' 12' 15'	6 13 15 3
	1' 7' 12' 3	1 4' 9' 6	4 7 2' 13	9 2 12 15	6' 13' 15' 3'
VIII	1 6 13 15	1' 3 10 14	3' 6' 4 9	10' 4' 13' 2	14' 9' 2' 15'
	1' 6' 13' 15'	1 3' 10' 14'	3 6 4' 9'	10 4 13 2'	14 9 2 15
IX	1 5 11 4'	1' 2 8 7'	2' 5' 3 12'	8' 3' 11' 15'	7 12 15 4
	1' 5' 11' 4'	1 2' 8' 7	2 5 3' 12	8 3 11 15	7' 12' 15' 4'
X	1 5 12 3'	1' 2 9 6'	2' 5' 4 11'	9' 4' 12' 15'	6 11 15 3
	1' 5' 12' 3	1 2' 9' 6	2 5 4' 11'	9 4 12 15	6' 11' 15' 3'
XI	1 6 11 15	1' 3 8 14	3' 6' 2 9	8' 2' 11' 4	14' 9' 4' 15'
	1' 6' 11' 15'	1 3' 8' 14'	3 6 2' 9'	8 2 11 4'	14 9 4 15
XII	1 7 12 15	1' 4 9 14	4' 7' 2 8	9' 2' 12' 3	14' 8' 3' 15'
	1' 7' 12' 15'	1 4' 9' 14'	4 7 2' 8'	9 2 12 3'	14 8 3 15

Regulators and characteristics corresponding to the Ist *type of parallelohedra.*

I	0 1 5 11 15	–1 0 0 0	λ_1	1–1 0 0	0 1 –1 0	0 0 1 –1	0 0 0 1
		2 1 1 1		(2) μ_1	(6) μ_4	(12) μ_6	(4′) λ_4
II	0 1 5 3′ 10′	–1 0 1 0	μ_2	1–1 0 0	0 1 0 0	0 0 –1 1	0 0 0 1
		1 0 –1 0		(2) μ_1	(14′) λ_2	(4′) μ_6	(12) λ_4
III	0 1 6 11 4′	–1 0 0 1	μ_3	1 0 –1 0	0 –1 1 0	0 1 0 0	0 0 0 –1
		1 0 0 –1		(3) μ_2	(5) μ_4	(9′) λ_2	(15) λ_4
IV	0 1 7 13 15	–1 0 0 0	λ_1	1 0 0 –1	0 0 –1 1	0 –1 1 0	0 1 0 0
		2 1 1 1		(4) μ_3	(6) μ_6	(12) μ_4	(2′) λ_2
V	0 1 5 12 15	–1 0 0 0	λ_1	1–1 0 0	0 1 0 –1	0 0 –1 1	0 0 1 0
		2 1 1 1		(2) μ_1	(7) μ_5	(11) μ_6	(3′) λ_3
VI	0 1 5 4′ 10′	–1 0 0 1	μ_3	1–1 0 0	0 1 0 0	0 0 1 –1	0 0 –1 0
		1 0 0 –1		(2) μ_1	(14′) λ_2	(3′) μ_6	(11) λ_3
VII	0 1 7 12 3′	–1 0 1 0	μ_2	1 0 0 –1	0 –1 0 1	0 1 0 0	0 0 –1 0
		1 0 –1 0		(4) μ_3	(5) μ_5	(8′) λ_2	(15) λ_3
VIII	0 1 6 13 15	–1 0 0 0	λ_1	1 0 –1 0	0 0 1 –1	0 –1 0 1	0 1 0 0
		2 1 1 1		(3) μ_2	(7) μ_6	(11) μ_5	(2′) λ_2
IX	0 1 5 11 4′	–1 0 0 1	μ_3	1–1 0 0	0 1 –1 0	0 0 1 0	0 0 0 –1
		1 0 0 –1		(2) μ_1	(6) μ_4	(10′) λ_3	(15) λ_4
X	0 1 5 12 3′	–1 0 1 0	μ_2	1–1 0 0	0 1 0 –1	0 0 0 1	0 0 –1 0
		1 0 –1 0		(2) μ_1	(7) μ_5	(10′) λ_4	(15) λ_3
XI	0 1 6 11 15	–1 0 0 0	λ_1	1 0 –1 0	0 –1 1 0	0 1 0 –1	0 0 0 1
		2 1 1 1		(3) μ_2	(5) μ_4	(13) μ_5	(4′) λ_4
XII	0 1 7 12 15	–1 0 0 0	λ_1	1 0 0 –1	0 –1 0 1	0 1 –1 0	0 0 1 0
		2 1 1 1		(4) μ_3	(5) μ_5	(13) μ_4	(3′) λ_3

In this table, the first line of each group contains the characteristics of faces in three dimensions corresponding to the simplexes $I, II, \ldots, XII$.

The second line contains the vertices of simplexes which are contiguous to the simplexes $I, II, \ldots, XII$ by the faces, the characteristic of which are indicated above in the first line, and the regulators are indicated near by in the second line.

115 Let us examine the parallelohedra in four dimensions which belong to the second type of primitive parallelohedra defined by the domain Δ' of quaternary quadratic forms. The domain Δ' is contiguous to the principal domain Δ by the face in a dimensions defined by the equation

$$\mu_1 = 0.$$

The independent regulator μ_1 corresponds to the faces of simplexes

$$I,\ II,\ V,\ VI,\ IX,\ ,X.$$

All these simplexes have to be reconstructed with the help of the algorithm explained in Number 91.

One will determine the numbers $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ after the conditions

$$(2) = \vartheta_1(1) + \vartheta_2(5) + \vartheta_3(11) + \vartheta_4(15) \text{ and } \vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4 = 1;$$

it becomes

$$\vartheta_0 = 1,\ \vartheta_1 = -1,\ \vartheta_2 = 1,\ \vartheta_3 = 0,\ \vartheta_4 = 0.$$

It follows that one will replace the three pairs of simplexes

$$(0,1,5,11,15) \text{ and } (0,2,5,11,15),$$
$$(0,1,5,12,15) \text{ and } (0,2,5,12,15),$$
$$(0,1,5,11,4') \text{ and } (0,2,5,11,4')$$

by the simplexes

$$\begin{array}{ccc} (0,1,5,11,15) & \text{and} & (0,1,2,11,15), \\ (2,1,5,12,15) & \text{and} & (0,1,2,12,15), \\ (2,1,5,11,4') & \text{and} & (0,1,2,11,4'). \end{array} \tag{3}$$

By designating the system $(1,-1,0,0)$ by the symbol (5) and the system $(-1,1,0,0)$ by the symbol $(5')$, one will designate

$$\text{I} - (0,5,1,6,13),\quad \text{II} - (0,1,2,11,15),\quad \text{V} - (0,5,1,7,13),$$
$$\text{VI} - (0,1,2,12,15),\quad \text{IX} - (0,5,1,6,9'),\quad \text{X} - (0,1,2,11,4').$$

These simplexes are congruent to the new simplexes (3).

The primitive parallelohedra of the II^{nd} type possess 120 vertices which are brought together in the following table:

Vertices of the primitive parallelohedron of II^{nd} *type*

Vertices of primitive parallelohedron of I^{st} *type*

I	$1\ 5\ 6\ 13$	$1'\ 2'\ 3\ 10$	$2\ 5'\ 8\ 14$	$3'\ 8'\ 6'\ 4$	$10'\ 14'\ 4'\ 13'$
	$1'\ 5'\ 6'\ 13'$	$1\ 2\ 3'\ 10'$	$2'\ 5\ 8'\ 14'$	$3\ 8\ 6\ 4'$	$10\ 14\ 4\ 13$
II	$1\ 2\ 11\ 15$	$1'\ 5'\ 8\ 14$	$5\ 2'\ 6\ 13$	$8'\ 6'\ 11'\ 4$	$14'\ 13'\ 4'\ 15'$
	$1'\ 2'\ 11'\ 15'$	$1\ 5\ 8'\ 14'$	$5'\ 2\ 6'\ 13'$	$8\ 6\ 11\ 4'$	$14\ 13\ 4\ 15$
III	$1\ 6\ 11\ 4'$	$1'\ 3\ 8\ 7'$	$3'\ 6'\ 2\ 13'$	$8'\ 2'\ 11'\ 15'$	$7\ 13\ 15\ 4$
	$1'\ 6'\ 11'\ 4$	$1\ 3'\ 8'\ 7$	$3\ 6\ 2'\ 13$	$8\ 2\ 11\ 15$	$7'\ 13'\ 15'\ 4'$
IV	$1\ 7\ 13\ 15$	$1'\ 4\ 10\ 14$	$4'\ 7'\ 3\ 8$	$10'\ 3'\ 13'\ 2$	$14'\ 8'\ 2'\ 15'$
	$1'\ 7'\ 13'\ 15'$	$1\ 4'\ 10'\ 14'$	$4\ 7\ 3'\ 8'$	$10\ 3\ 13\ 2'$	$14\ 8\ 2\ 15$
V	$1\ 5\ 7\ 13$	$1'\ 2'\ 4\ 10$	$2\ 5'\ 9\ 14$	$4'\ 9'\ 7'\ 3$	$10'\ 14'\ 3'\ 13'$
	$1'\ 5'\ 7'\ 13'$	$1\ 2\ 4'\ 10'$	$2'\ 5\ 9'\ 14'$	$4\ 9\ 7\ 3'$	$10\ 14\ 3\ 13$
VI	$1\ 2\ 12\ 15$	$1'\ 5'\ 9\ 14$	$5\ 2'\ 7\ 13$	$9'\ 7'\ 12'\ 3$	$14'\ 13'\ 3'\ 15'$
	$1'\ 2'\ 12'\ 15'$	$1\ 5\ 9'\ 14'$	$5'\ 2\ 7'\ 13'$	$9\ 7\ 12\ 3'$	$14\ 13\ 3\ 15$
VII	$1\ 7\ 12\ 3'$	$1'\ 4\ 9\ 6'$	$4'\ 7'\ 2\ 13'$	$9'\ 2'\ 12'\ 15'$	$6\ 13\ 15\ 3$
	$1'\ 7'\ 12'\ 3$	$1\ 4'\ 9'\ 6$	$4\ 7\ 2'\ 13$	$9\ 2\ 12\ 15$	$6'\ 13'\ 15'\ 3'$
VIII	$1\ 6\ 13\ 15$	$1'\ 3\ 10\ 14$	$3'\ 6'\ 4\ 9$	$10'\ 4'\ 13'\ 2$	$14'\ 9'\ 2'\ 15'$
	$1'\ 6'\ 13'\ 15'$	$1\ 3'\ 10'\ 14'$	$3\ 6\ 4'\ 9'$	$10\ 4\ 13\ 2'$	$14\ 9\ 2\ 15$
IX	$1\ 5\ 6\ 9'$	$1'\ 2'\ 3\ 12'$	$2\ 5'\ 8\ 7'$	$3'\ 8'\ 6'\ 15'$	$12\ 7\ 15\ 9$
	$1'\ 5'\ 6'\ 9$	$1\ 2\ 3'\ 12$	$2'\ 5\ 8'\ 7$	$3\ 8\ 6\ 15$	$12'\ 7'\ 15'\ 9'$
X	$1\ 2\ 11\ 4'$	$1'\ 5'\ 8\ 7'$	$5\ 2'\ 6\ 9'$	$8'\ 6'\ 11'\ 15'$	$7\ 9\ 15\ 4$
	$1'\ 2'\ 11'\ 4$	$1\ 5\ 8'\ 7$	$5'\ 2\ 6'\ 9$	$8\ 6\ 11\ 15$	$7'\ 9'\ 15'\ 4'$
XI	$1\ 6\ 11\ 15$	$1'\ 3\ 8\ 14$	$3'\ 6'\ 2\ 9$	$8'\ 2'\ 11'\ 4$	$14'\ 9'\ 4'\ 15'$
	$1'\ 6'\ 11'\ 15'$	$1\ 3'\ 8'\ 14'$	$3\ 6\ 2'\ 9'$	$8\ 2\ 11\ 4'$	$14\ 9\ 4\ 15$
XII	$1\ 7\ 12\ 15$	$1'\ 4\ 9\ 14$	$4'\ 7'\ 2\ 8$	$9'\ 2'\ 12'\ 3$	$14'\ 8'\ 3'\ 15'$
	$1'\ 7'\ 12'\ 15'$	$1\ 4'\ 9'\ 14'$	$4\ 7\ 2'\ 8'$	$9\ 2\ 12\ 3'$	$14\ 8\ 3\ 15$

Regulators and characteristic corresponding to the IInd-type of parallelohedra

I	$0\ 1\ 5\ 6\ 13$	$-1\ 0\ 0\ 0$ $2\ 0\ 1\ 1$	λ_1	$1\ 1\ -1\ 0$ $(2')\ \mu_1$	$0\ -1\ 0\ 0$ $(15)\ \lambda_2$	$0\ 0\ 1\ -1$ $(7)\ \mu_6$	$0\ 0\ 0\ 1$ $(9')\ \lambda_4+\mu_1$
II	$0\ 1\ 2\ 11\ 15$	$-1\ -1\ 1\ 0$ $1\ 1\ 0\ 0$	μ_1	$1\ 0\ -1\ 0$ $(8)\ \mu_2$	$0\ 1\ -1\ 0$ $(6)\ \mu_4$	$0\ 0\ 1\ -1$ $(12)\ \mu_6+\mu_1$	$0\ 0\ 0\ 1$ $(4')\ \lambda_4$
III	$0\ 1\ 6\ 11\ 4'$	$-1\ 0\ 0\ 1$ $1\ 0\ 0\ -1$	$\mu_3+\mu_1$	$1\ 0\ -1\ 0$ $(8)\ \mu_2$	$0\ -1\ 1\ 0$ $(2)\ \mu_4$	$0\ 1\ 0\ 0$ $(9')\ \lambda_2+\mu_1$	$0\ 0\ 0\ -1$ $(15)\ \lambda_4$
IV	$0\ 1\ 7\ 13\ 15$	$-1\ 0\ 0\ 0$ $2\ 0\ 1\ 1$	λ_1	$1\ 0\ 0\ -1$ $(4)\ \mu_3+\mu_1$	$0\ 0\ -1\ 1$ $(6)\ \mu_6$	$0\ -1\ 1\ 0$ $(12)\ \mu_4+\mu_1$	$0\ 1\ 0\ 0$ $(5)\ \lambda_2$
V	$0\ 1\ 5\ 7\ 13$	$-1\ 0\ 0\ 0$ $2\ 0\ 1\ 1$	λ_1	$1\ 1\ 0\ -1$ $(2')\ \mu_1$	$0\ -1\ 0\ 0$ $(15)\ \lambda_2$	$0\ 0\ -1\ 1$ $(6)\ \mu_6$	$0\ 0\ 1\ 0$ $(8')\ \lambda_3+\mu_1$
VI	$0\ 1\ 2\ 12\ 15$	$-1\ -1\ 0\ 1$ $1\ 1\ 0\ 0$	μ_1	$1\ 0\ 0\ -1$ $(9)\ \mu_3$	$0\ 1\ 0\ -1$ $(7)\ \mu_5$	$0\ 0\ -1\ 1$ $(11)\ \mu_6+\mu_1$	$0\ 0\ 1\ 0$ $(3')\ \lambda_3$
VII	$0\ 1\ 7\ 12\ 3'$	$-1\ 0\ 1\ 0$ $1\ 0\ -1\ 0$	$\mu_2+\mu_1$	$1\ 0\ 0\ -1$ $(9)\ \mu_3$	$0\ -1\ 0\ 1$ $(2)\ \mu_5$	$0\ 1\ 0\ 0$ $(8')\ \lambda_2+\mu_1$	$0\ 0\ -1\ 0$ $(15)\ \lambda_3$
VIII	$0\ 1\ 6\ 13\ 15$	$-1\ 0\ 0\ 0$ $2\ 0\ 1\ 1$	λ_1	$1\ 0\ -1\ 0$ $(3)\ \mu_2+\mu_1$	$0\ 0\ 1\ -1$ $(7)\ \mu_6$	$0\ -1\ 0\ 1$ $(11)\ \mu_5+\mu_1$	$0\ 1\ 0\ 0$ $(5)\ \lambda_2$
IX	$0\ 1\ 5\ 6\ 9'$	$-1\ 0\ 0\ 1$ $1\ -1\ 0\ -1$	μ_3	$1\ 1\ -1\ -1$ $(2')\ \mu_1$	$0\ -1\ 0\ 1$ $(4')\ \mu_5$	$0\ 0\ 1\ 0$ $(14')\ \lambda_3$	$0\ 0\ 0\ -1$ $(13)\ \lambda_4+\mu_1$
X	$0\ 1\ 2\ 11\ 4'$	$-1\ -1\ 1\ 1$ $1\ 1\ 0\ 0$	μ_1	$1\ 0\ -1\ 0$ $(8)\ \mu_2$	$0\ 1\ -1\ 0$ $(6)\ \mu_4$	$0\ 0\ 1\ 0$ $(10')\ \lambda_3+\mu_1$	$0\ 0\ 0\ -1$ $(15)\ \lambda_4$
XI	$0\ 1\ 6\ 11\ 15$	$-1\ 0\ 0\ 0$ $2\ 1\ 1\ 1$	$\lambda_1+\mu_1$	$1\ 0\ -1\ 0$ $(8)\ \mu_2$	$0\ -1\ 1\ 0$ $(2)\ \mu_4$	$0\ 1\ 0\ -1$ $(13)\ \mu_5+\mu_1$	$0\ 0\ 0\ 1$ $(4')\ \lambda_4$
XII	$0\ 1\ 7\ 12\ 15$	$-1\ 0\ 0\ 0$ $2\ 1\ 1\ 1$	$\lambda_1+\mu_1$	$1\ 0\ 0\ -1$ $(9)\ \mu_3$	$0\ -1\ 0\ 1$ $(2)\ \mu_5$	$0\ 1\ -1\ 0$ $(13)\ \mu_4+\mu_1$	$0\ 0\ 1\ 0$ $(3')\ \lambda_3$

The domain Δ' of quaternary quadratic forms which define the second type of primitive parallelohedra in four dimensions is determined by 10 independent inequalities

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0,$$

$$\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \mu_4 \geq 0, \mu_5 \geq 0, \mu_6 \geq 0.$$

Any quaternary quadratic form belonging to the domain Δ' can be written

$$f(x_1,x_2,x_3,x_4) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \mu_1 \omega + \mu_2 (x_1 - x_3)^2 +$$
$$\mu_3 (x_1 - x_4)^2 + \mu_4 (x_2 - x_3)^2 + \mu_5 (x_2 - x_4)^2 + \mu_6 (x_3 - x_4)^2$$

where

$$\omega = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 + 2x_1x_2 - 2x_1x_3 - 2x_1x_4 - 2x_2x_3 - 2x_2x_4. \tag{4}$$

The parallelohedra belonging to the IInd type are determined by 30 inequalities of form (2) which are symbolically presented in the following table:

IInd*-type of parallelohedra*

N	l_1	l_2	l_3	l_4	$f(l_1, l_2, l_3, l_4)$	$-l_1$	$-l_2$	$-l_3$	$-l_4$	N
1	1	0	0	0	$\lambda_1 + 2\mu_1 + \mu_2 + \mu_3$	-1	0	0	0	$1'$
2	0	1	0	0	$\lambda_2 + 2\mu 1_{+}\mu_4 + \mu_5$	0	-1	0	0	$2'$
3	0	0	1	0	$\lambda_3 + 2\mu_1 + \mu_2 + \mu_4 + \mu_6$	0	0	-1	0	$3'$
4	0	0	0	1	$\lambda_4 + 2\mu_1 + \mu_3 + \mu_5 + \mu_6$	0	0	0	-1	$4'$
5	1	-1	0	0	$\lambda_1 + \lambda_2 + 2\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	-1	1	0	0	$5'$
6	1	0	1	0	$\lambda_1 + \lambda_3 + 2\mu_1 + \mu_3 + \mu_4 + \mu_6$	-1	0	-1	0	$6'$
7	1	0	0	1	$\lambda_1 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_5 + \mu_6$	-1	0	0	-1	$7'$
8	0	1	1	0	$\lambda_2 + \lambda_3 + 2\mu_1 + \mu_2 + \mu_5 + \mu_6$	0	-1	-1	0	$8'$
9	0	1	0	1	$\lambda_2 + \lambda_4 + 2\mu_1 + \mu_3 + \mu_4 + \mu_6$	0	-1	0	-1	$9'$
10	0	0	1	1	$\lambda_3 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	0	0	-1	-1	$10'$
11	1	1	1	0	$\lambda_1 + \lambda_2 + \lambda_3 + 4\mu_1 + \mu_3 + \mu_5 + \mu_6$	-1	-1	-1	0	$11'$
12	1	1	0	1	$\lambda_1 + \lambda_2 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_4 + \mu_6$	-1	-1	0	-1	$12'$
13	1	0	1	1	$\lambda_1 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_4 + \mu_5$	-1	0	-1	-1	$13'$
14	0	1	1	1	$\lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_3$	0	-1	-1	-1	$14'$
15	1	1	1	1	$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1$	-1	-1	-1	-1	$15'$

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Let us find the domain Δ'' of quaternary quadratic forms which is contiguous to the domain Δ' by the face in 9 dimensions defined by the equation

$$\mu_6 = 0.$$

The independent regulator μ_6 corresponds to the faces of simplexes

$$I,\ IV,\ V,\ VIII.$$

All these simplexes have to be reconstructed after the algorithm explained in Number 91. One will determine, to this effect, the number

$$\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$$

after the condition

$$\vartheta_1(1) + \vartheta_2(5) + \vartheta_3(6) + \vartheta_4(13) \quad \text{and} \quad \vartheta_0 + \vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4; \tag{5}$$

it becomes

$$\vartheta_0 = 0, \quad \vartheta_1 = 1, \quad \vartheta_2 = 0, \quad \vartheta_3 = -1, \quad \vartheta_4 = 1.$$

One concludes that the two pairs of simplexes

$$(0,1,5,6,13) \text{ and } (0,1,5,7,13) \text{ and}$$
$$(0,1,7,13,15) \text{ and } (0,1,6,13,15)$$

have to be replaced by the new simplexes

$$\text{I} - (0,7,5,6,13) \quad \text{and} \quad \text{V} - (0,1,5,6,7)$$
$$\text{IV} - (0,6,7,13,15) \quad \text{and} \quad \text{VIII} - (0,1,7,6,15).$$

By designating the system $(0,0,1,-1)$ by the symbol (10) and the system $(0,0,-1,1)$ by the symbol $(10')$ one will determine all the vertices of primitive parallelohedra belonging to the new type as follows.

Vertices of primitive parallelohedron of III^{rd}-type.

I	$7\ 5\ 6\ 13$	$7'\ 9'\ 10\ 3$	$9\ 5'\ 8\ 14$	$10'\ 8'\ 6'\ 4$	$3'\ 14'\ 4'\ 13'$
	$7'\ 5'\ 6'\ 13'$	$7\ 9\ 10'\ 3'$	$9'\ 5\ 8'\ 14'$	$10\ 8\ 6\ 4'$	$3\ 14\ 4\ 13$
II	$1\ 2\ 11\ 15$	$1'\ 5'\ 8\ 14$	$5\ 2'\ 6\ 13$	$8'\ 6'\ 11'\ 4$	$14'\ 13'\ 4'\ 15'$
	$1'\ 2'\ 11'\ 15'$	$1\ 5\ 8'\ 14'$	$5'\ 2\ 6'\ 13'$	$8\ 6\ 11\ 4'$	$14\ 13\ 4\ 15$
III	$1\ 6\ 11\ 4'$	$1'\ 3\ 8\ 7'$	$3'\ 6'\ 2\ 13'$	$8'\ 2'\ 11'\ 15'$	$7\ 13\ 15\ 4$
	$1'\ 6'\ 11'\ 4$	$1\ 3'\ 8'\ 7$	$3\ 6\ 2'\ 13$	$8\ 2\ 11\ 15$	$7'\ 13'\ 15'\ 4'$
IV	$6\ 7\ 13\ 15$	$6'\ 10'\ 4\ 9$	$10\ 7'\ 3\ 8$	$4'\ 3'\ 13'\ 2$	$9'\ 8'\ 2'\ 15'$
	$6'\ 7'\ 13'\ 15'$	$6\ 10\ 4'\ 9'$	$10'\ 7\ 3'\ 8'$	$4\ 3\ 13\ 2'$	$9\ 8\ 2\ 15$
V	$1\ 5\ 6\ 7$	$1'\ 2'\ 3\ 4$	$2\ 5'\ 8\ 9$	$3'\ 8'\ 6'\ 10'$	$4'\ 9'\ 10\ 7'$
	$1'\ 5'\ 6'\ 7'$	$1\ 2\ 3'\ 4'$	$2'\ 5\ 8'\ 9'$	$3\ 8\ 6\ 10$	$4\ 9\ 10'\ 7$
VI	$1\ 2\ 12\ 15$	$1'\ 5'\ 9\ 14$	$5\ 2'\ 7\ 13$	$9'\ 7'\ 12'\ 3$	$14'\ 13'\ 3'\ 15'$
	$1'\ 2'\ 12'\ 15'$	$1\ 5\ 9'\ 14'$	$5'\ 2\ 7'\ 13'$	$9\ 7\ 12\ 3'$	$14\ 13\ 3\ 15$
VII	$1\ 7\ 12\ 3'$	$1'\ 4\ 9\ 6'$	$1\ 4'\ 9'\ 6$	$4'\ 7'\ 2\ 13'$	$9'\ 2'\ 12'\ 15'$
	$6\ 13\ 15\ 3$	$1'\ 7'\ 12'\ 3$	$4\ 7\ 2\ 13$	$9\ 2\ 12\ 15$	$6'\ 13'\ 15'\ 3'$
VIII	$1\ 7\ 6\ 15$	$1'\ 4\ 3\ 14$	$4'\ 7'\ 10\ 8$	$3'\ 10'\ 6'\ 9$	$14'\ 8'\ 9'\ 15'$
	$1'\ 7'\ 6'\ 15'$	$1\ 4'\ 3'\ 14'$	$4\ 7\ 10'\ 8'$	$3\ 10\ 6\ 9'$	$14\ 8\ 9\ 15$
IX	$1\ 5\ 6\ 9'$	$1'\ 2'\ 3\ 12'$	$2\ 5'\ 8\ 7'$	$3'\ 8'\ 6'\ 15'$	$12\ 7\ 15\ 9$
	$1'\ 5'\ 6'\ 9$	$1\ 2\ 3'\ 12$	$2'\ 5\ 8'\ 7$	$3\ 8\ 6\ 15$	$12'\ 7'\ 15'\ 9'$
X	$1\ 2\ 11\ 4'$	$1'\ 5'\ 8\ 7'$	$5\ 2'\ 6\ 9'$	$8'\ 6'\ 11'\ 15'$	$7\ 9\ 15\ 4$
	$1'\ 2'\ 11'\ 4$	$1\ 5\ 8'\ 7$	$5'\ 2\ 6'\ 9$	$8\ 6\ 11\ 15$	$7'\ 9'\ 15'\ 4'$
XI	$1\ 6\ 11\ 15$	$1'\ 3\ 8\ 14$	$3'\ 6'\ 2\ 9$	$8'\ 2'\ 11'\ 4$	$14'\ 9'\ 4'\ 15'$
	$1'\ 6'\ 11'\ 15'$	$1\ 3'\ 8'\ 14'$	$3\ 6\ 2'\ 9'$	$8\ 2\ 11\ 4'$	$14\ 9\ 4\ 14$
XII	$1\ 7\ 12\ 15$	$1'\ 4\ 9\ 14$	$4'\ 7'\ 2\ 8$	$9'\ 2'\ 12'\ 3$	$14'\ 8'\ 3'\ 15'$
	$1'\ 7'\ 12'\ 15'$	$1\ 4'\ 9'\ 14'$	$4\ 7\ 2'\ 8'$	$9\ 2\ 12\ 3'$	$14\ 8\ 3\ 15$

Regulators and characteristics corresponding to the IIIrd-type parallelohedra

I	$0\ 7\ 5\ 6\ 13$	$-1\ 0\ 0\ 0$ $2\ 0\ 1\ 1$	λ_1	$1\ 1\ -1\ 0$ $(2')\ \mu_1$	$0\ -1\ 0\ 0$ $(15)\ \lambda_2$	$1\ 1\ 0\ -1$ $(2')\ \mu_1$	$-1\ -1\ 1\ 1$ $(1)\ \mu_6$
II	$0\ 1\ 2\ 11\ 15$	$-1\ -1\ 1\ 0$ $1\ 1\ 0\ 1$	μ_1	$1\ 0\ -1\ 0$ $(8)\ \mu_2$	$0\ 1\ -1\ 0$ $(6)\ \mu_4$	$0\ 0\ 1\ -1$ $(12)\ \mu_1$	$0\ 0\ 0\ 1$ $(4')\ \lambda_4$
III	$0\ 1\ 6\ 11\ 4'$	$-1\ 0\ 0\ 1$ $1\ 0\ 1\ -1$	$\mu_3+\mu_1$	$1\ 0\ -1\ 0$ $(8)\ \mu_2$	$0\ -1\ 1\ 0$ $(2)\ \mu_4$	$0\ 1\ 0\ 0$ $(9')\ \lambda_2+\mu_1$	$0\ 0\ 0\ -1$ $(15)\ \lambda_4$
IV	$0\ 6\ 7\ 13\ 15$	$-1\ 0\ 0\ 0$ $2\ 0\ 1\ 1$	λ_1	$1\ 0\ 0\ -1$ $(4)\ \mu_3+\mu_1$	$1\ 0\ -1\ 0$ $(3)\ \mu_2+\mu_1$	$-1\ -1\ 1\ 1$ $(1)\ \mu_6$	$0\ 1\ 0\ 0$ $(5)\ \lambda_2$
V	$0\ 1\ 5\ 6\ 7$	$-1\ 0\ 0\ 0$ $2\ 0\ 1\ 1$	λ_1	$1\ 1\ -1\ -1$ $(13)\ \mu_6$	$0\ -1\ 0\ 0$ $(15)\ \lambda_2$	$0\ 0\ 1\ 0$ $(8')\ \lambda_3+\mu_1$	$0\ 0\ 0\ 1$ $(9')\ \lambda_4+\mu_1$
VI	$0\ 1\ 2\ 12\ 15$	$-1\ -1\ 0\ 1$ $1\ 1\ 1\ 0$	μ_1	$1\ 0\ 0\ -1$ $(9)\ \mu_3$	$1\ 1\ 0\ -1$ $(7)\ \mu_5$	$0\ 0\ -1\ 1$ $(11)\ \mu_1$	$0\ 0\ 1\ 0$ $(3')\ \lambda_3$
VII	$0\ 1\ 7\ 12\ 3'$	$-1\ 0\ 0\ 0$ $1\ 0\ -1\ 1$	$\mu_2+\mu_1$	$1\ 0\ 0\ -1$ $(9)\ \mu_3$	$0\ -1\ 0\ 1$ $(2)\ \mu_5$	$0\ 1\ 0\ 0$ $(8')\ \lambda_2+\mu_1$	$0\ 0\ -1\ 0$ $(15)\ \lambda_3$
VIII	$0\ 1\ 7\ 6\ 15$	$-1\ 0\ 0\ 0$ $2\ 0\ 1\ 1$	λ_1	$1\ 1\ -1\ -1$ $(13)\ \mu_6$	$0\ -1\ 0\ 1$ $(11)\ \mu_5+\mu_1$	$0\ -1\ 1\ 0$ $(12)\ \mu_4+\mu_1$	$0\ 1\ 0\ 0$ $(5)\ \lambda_2$
IX	$0\ 1\ 5\ 6\ 9'$	$-1\ 0\ 0\ 1$ $1\ -1\ 0\ -1$	μ_3	$1\ 1\ -1\ -1$ $(2')\ \mu_1+\mu_6$	$0\ -1\ 0\ 1$ $(4')\ \mu_5$	$0\ 0\ 1\ 0$ $(14')\ \lambda_3$	$0\ 0\ 0\ -1$ $(7)\ \lambda_4+\mu_1$
X	$0\ 1\ 2\ 11\ 4'$	$-1\ -1\ 1\ 1$ $1\ 1\ 0\ 0$	$\mu_1+\mu_6$	$1\ 0\ -1\ 0$ $(8)\ \mu_2$	$0\ 1\ -1\ 0$ $(6)\ \mu_4$	$0\ 0\ 1\ 0$ $(3')\ \lambda_3+\mu_1$	$0\ 0\ 0\ -1$ $(15)\ \lambda_4$
XI	$0\ 1\ 6\ 11\ 15$	$-1\ 0\ 0\ 0$ $2\ 1\ 1\ 1$	$\lambda_1+\mu_1$	$1\ 0\ -1\ 0$ $(8)\ \mu_2$	$0\ -1\ 1\ 0$ $(2)\ \mu_4$	$0\ 1\ 0\ -1$ $(7)\ \mu_5+\mu_1$	$0\ 0\ 0\ 1$ $(4')\ \lambda_4$
XII	$0\ 1\ 7\ 12\ 15$	$-1\ 0\ 0\ 0$ $2\ 1\ 1\ 1$	$\lambda_1+\mu_1$	$1\ 0\ 0\ -1$ $(9)\ \mu_3$	$0\ -1\ 0\ 1$ $(2)\ \mu_5$	$0\ 1\ -1\ 0$ $(6)\ \mu_4+\mu_1$	$0\ 0\ 1\ 0$ $(3')\ \lambda_3$

The independent regulators are expressed by the formulae

$$\lambda_1 = a_{11} + a_{13} + a_{14} + a_{34}, \lambda_2 = a_{22} + a_{23} + a_{24} + a_{34}, \lambda_3 = a_{31} + a_{32} + a_{33} + a_{34},$$

$$\lambda_4 = a_{41} + a_{42} + a_{43} + a_{44}, \mu_1 = a_{12} - a_{34}, \mu_2 = -a_{13} - a_{12},$$

$$\mu_3 = -a_{14} - a_{12}, \mu_4 = -a_{23} - a_{12}, \mu_5 = -a_{24} - a_{12}, \mu_6 = a_{34}.$$

The domain Δ'' of quaternary quadratic forms which defines the third type of primitive parallelohedra in four dimensions is determined by 10 independent inequalities

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0,$$

$$\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \mu_4 \geq 0, \mu_5 \geq 0, \mu_6 \geq 0.$$

Each quaternary quadratic form belonging to the domain Δ'' can be written

$$\begin{aligned} f(x_1, x_2, x_3, x_4) &= \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \mu_1 \omega + \mu_2 (x_1 - x_3)^2 \\ &+ \mu_3 (x_1 - x_4)^2 + \mu_4 (x_2 - x_3)^2 + \mu_5 (x_2 - x_4)^2 + \mu_6 (x_1 + x_2 - x_3 - x_4)^2, \end{aligned}$$

the form ω being defined by the equality (4). The parallelohedra belonging to the IIIrd type are determined by 30 inequalities of the form (2) which are symbolically presented in the following table:

IIIrd type of parallelohedra

N	$l_1\ l_2 l_3 l_4$	$f(l_1, l_2, l_3, l_4)$	$-l_1 - l_2 - l_3 - l_4$	N
1	1 0 0 0	$\lambda_1 + 2\mu_1 + \mu_2 + \mu_3 + \mu_6$	−1 0 0 0	1′
2	0 1 0 0	$\lambda_2 + 2\mu_1 + \mu_4 + \mu_5 + \mu_6$	0 −1 0 0	2′
3	0 0 1 0	$\lambda_3 + 2\mu_1 + \mu_2 + \mu_4 + \mu_6$	0 0 −1 0	3′
4	0 0 0 1	$\lambda_4 + 2\mu_1 + \mu_3 + \mu_5 + \mu_6$	0 0 0 −1	4′
5	1 −1 0 0	$\lambda_1 + \lambda_2 + 2\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	−1 1 0 0	5′
6	1 0 1 0	$\lambda_1 + \lambda_3 + 2\mu_1 + \mu_3 + \mu_4$	−1 0 −1 0	6′
7	1 0 0 1	$\lambda_1 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_5$	−1 0 0 −1	7′
8	0 1 1 0	$\lambda_2 + \lambda_3 + 2\mu_1 + \mu_2 + \mu_5$	0 −1 −1 0	8′
9	0 1 0 1	$\lambda_2 + \lambda_4 + 2\mu_1 + \mu_3 + \mu_4$	0 −1 0 −1	9′
10	0 0 1 −1	$\lambda_3 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	0 0 −1 1	10′
11	1 1 1 0	$\lambda_1 + \lambda_2 + \lambda_3 + 4\mu_1 + \mu_3 + \mu_5 + \mu_6$	−1 −1 −1 0	11′
12	1 1 0 1	$\lambda_1 + \lambda_2 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_4 + \mu_6$	−1 −1 0 −1	12′
13	1 0 1 1	$\lambda_1 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_4 + \mu_5 + \mu_6$	−1 0 −1 −1	13′
14	0 1 1 1	$\lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_3 + \mu_6$	0 −1 −1 −1	14′
15	1 1 1 1	$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1$	−1 −1 −1 −1	15′

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We have determined three domains $\Delta, \Delta', \Delta''$ which characterise three types of primitive parallelohedra in four dimensions.

Theorem. The set (Δ) *of domains of quaternary quadratic forms is composed of three different classes which can be represented by the domain* $\Delta, \Delta', \Delta''$.

In my first mémoire cited, it has been demonstrated that the set (R) of domains of quaternary quadratic forms corresponding to the perfect quaternary quadratic forms is composed of two classes represented by the principal domain R and by a domain R, determined by the equalities

$$\begin{aligned} f(x_1,x_2,x_3,x_4) = \rho_1 x_1^2 + \rho_2 x_2^2 + \rho_3 x_3^2 + \rho_4 x_4^2 + \rho_5 (x_1 - x_3)^2 + \\ \rho_6 (x_1 - x_4)^2 + \rho_7 (x_1 - x_3)^2 + \rho_8 (x_2 - x_4)^2 + \rho_9 (x_3 - x_4)^2 + \\ \rho_{10} (x_1 + x_2 - x_3)^2 + \rho_{11} (x_1 + x_2 - x_4)^2 + \rho_{12} (x_1 + x_2 - x_3 - x_4)^2 \end{aligned}$$

where $\rho_1, \rho_2, \ldots, \rho_{12}$ are positive arbitrary parameters or zeros.

The domain R, corresponds to a perfect form

$$\varphi_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4. \tag{5}$$

In the mémoire cited, it has been demonstrated that all the faces in 9 dimensions of domain R, are equivalent to two faces characterised: one by the quadratic form

$$x_1^2, x_2^2, x_3^2, x_4^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_2 - x_3)^2, (x_2 - x_4)^2, (x_3 - x_4)^2$$

and the other by the quadratic form

$$\begin{gathered} x_1^2, x_2^2, x_3^2, x_4^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_2 - x_3)^2, (x_2 - x_4)^2, \\ (x_1 + x_2 - x_3 - x_4)^2. \end{gathered}$$

The first face verifies the equation

$$a_{12} = 0$$

and the second face verifies the equation

$$a_{12} - a_{34} = 0$$

The form ω determined by the formula (4) characterise the axis of the domain (R) which does not change when one transforms the domain R into itself.

One concludes that the domain R can be partitioned into groups all of which are equivalent to the two domains Δ' and Δ'' obtained. This results in that the principal domain Δ and the two domains Δ' and Δ'' can not be equivalent.

The theorem introduced is thus demonstrated.

By not considering as different the equivalent types of parallelohedra one can say that there are only three different types of primitive parallelohedra in the space in four dimensions.

By calling reduced the positive quadratic forms which belong to the domains Δ, Δ' and Δ'' one obtains a new method of reduction of quaternary positive quadratic forms which presents a modification of the method due to Mr. Charve. ‡

In effect, following the method of Mr. Charve, one calls reduced the quaternary positive quadratic forms belonging to one of three simple domains R, R' and R''. The first two domain R and R' coincide with the domains Δ and Δ' and it is only the third domain R'' of Mr. Charve which differs from the domain Δ''. Any form belonging to the domain Δ'' is equivalent to a form belonging to the domain R'' and vice-versa.

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By examining the two tables which contain the characteristics of faces of simplexes which define the 2nd and the 3rd type of primitive parallelohedra, one will observe that these characteristic coincide for the two types and are represented by the linear forms

$$\begin{gathered} \pm x_1, \pm x_2, \pm x_3, \pm x_4, \pm(x_1 - x_3), \pm(x_1 - x_4), \pm(x_2 - x_3), \pm(x_2 - x_4), \pm(x_3 - x_4), \\ \pm(x_1 + x_2 - x_3), \pm(x_1 + x_2 - x_4), \pm(x_1 + x_2 - x_3 - x_4). \end{gathered}$$

‡ See the introduction

It is remarkable that these linear forms define the set of representations of the minimum of the perfect form φ_1 determined by the equality (5).

By virtue of that which has previously been mentioned, one can affirm that the coincidence noticed appears as the characteristics of faces of all the primitive parallelohedra in 2, 3 and 4 dimensions.

It would be interesting to find out whether this is only a coincidence or whether there really exists a relation between the two problems which seem to be different: between the problem of the uniform partition of the space with the help of congruent convex polyhedra and the study of perfect positive quadratic forms.

End of the second mémoire

[in German]

Immediately after the first sheet of this significant work was set, we received the grievous tidings that your author of the science has been taken away by Death. The editor had in the best power seen to it that this last work of he who so early departed for the other side was checked over with utmost care.

Marburg, 19th June 1909

K. Hensel

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